11 Radicals and Geometry Connections

11.1 Graph Square Root Functions
11.2 Simplify Radical Expressions
11.3 Solve Radical Equations
11.4 Apply the Pythagorean Theorem and Its Converse
11.5 Apply the Distance and Midpoint Formulas

Before

In previous chapters, you learned the following skills, which you’ll use in Chapter 11: comparing the graphs of functions with the graphs of parent functions, evaluating square roots, using the distributive property, factoring trinomials, and evaluating expressions.

Prerequisite Skills

VOCABULARY CHECK
Copy and complete the statement.
1. The number or expression inside a radical symbol is called the ___.
2. If \( b^2 = a \), then \( b \) is a(n) ___ of \( a \).

SKILLS CHECK
1. Graph \( y = 3 \cdot 2^x \). Compare the graph with the graph of \( y = 2^x \). (Review p. 520 for 11.1.)

Evaluate the expression. (Review p. 110 for 11.2.)
4. \( \sqrt{81} \) 5. \(-\sqrt{64}\) 6. \( \pm\sqrt{100} \) 7. \(-\sqrt{121}\)

Use the distributive property to write an equivalent expression. (Review p. 96 for 11.2.)
8. \( 4(y - 3) \) 9. \( 2(x - 2) \) 10. \( -x(x + 11) \) 11. \( 4x(x - 9) \)

Factor the trinomial. (Review p. 583 for 11.3.)
12. \( x^2 + 4x + 4 \) 13. \( m^2 + 9m + 8 \) 14. \( r^2 + 8r + 7 \) 15. \( b^2 + 10b + 16 \)
16. Evaluate \( a^2 \) when \( a = 7 \). (Review p. 2 for 11.4–11.5.)

@HomeTutor Prerequisite skills practice at classzone.com
In Chapter 11, you will apply the big ideas listed below and reviewed in the Chapter Summary on page 753. You will also use the key vocabulary listed below.

**Big Ideas**

1. **Graphing square root functions**
2. **Using properties of radicals in expressions and equations**
3. **Working with radicals in geometry**

**Key Vocabulary**

- radical expression, p. 710
- radical function, p. 710
- square root function, p. 710
- parent square root function, p. 710
- simplest form of a radical expression, p. 719
- rationalizing the denominator, p. 721
- radical equation, p. 729
- extraneous solution, p. 730
- hypotenuse, p. 737
- legs of a right triangle, p. 737
- Pythagorean theorem, p. 737
- distance formula, p. 744
- midpoint, p. 745
- midpoint formula, p. 745

**Why?**

You can use radical equations to solve real-world problems. For example, you can find the length of a sailboat’s waterline given the hull speed of the sailboat.

**Animated Algebra**

The animation illustrated below for Example 5 on page 731 helps you answer this question: What is the length of a sailboat’s waterline if the sailboat has a hull speed of 8 nautical miles per hour?

You need to find the length of the sailboat’s waterline.

Other animations for Chapter 11: pages 711, 719, 722, 737, 746, and 753
### 11.1 Graph Square Root Functions

**Key Vocabulary**
- radical expression
- radical function
- square root function
- parent square root function

A **radical expression** is an expression that contains a radical, such as a square root, cube root, or other root. A **radical function** contains a radical expression with the independent variable in the radicand. For example, \( y = \sqrt{2x} \) and \( y = \sqrt{x} + 2 \) are radical functions. If the radical is a square root, then the function is called a **square root function**.

**Example 1**

**Graph a function of the form** \( y = a\sqrt{x} \)

Graph the function \( y = 3\sqrt{x} \) and identify its domain and range. Compare the graph with the graph of \( y = \sqrt{x} \).

**Solution**

1. **Step 1** Make a table. Because the square root of a negative number is undefined, \( x \) must be nonnegative. So, the domain is \( x \geq 0 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>0</td>
<td>3</td>
<td>4.2</td>
<td>5.2</td>
<td>6</td>
</tr>
</tbody>
</table>

2. **Step 2** Plot the points.

3. **Step 3** Draw a smooth curve through the points. From either the table or the graph, you can see the range of the function is \( y \geq 0 \).

4. **Step 4** Compare the graph with the graph of \( y = \sqrt{x} \). The graph of \( y = 3\sqrt{x} \) is a vertical stretch (by a factor of 3) of the graph of \( y = \sqrt{x} \).


**Example 2**  
**Graph a function of the form** $y = a\sqrt{x}$

Graph the function $y = -0.5\sqrt{x}$ and identify its domain and range. Compare the graph with the graph of $y = \sqrt{x}$.

**Solution**

To graph the function, make a table, plot the points, and draw a smooth curve through the points. The domain is $x \geq 0$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>0</td>
<td>-0.5</td>
<td>-0.7</td>
<td>-0.9</td>
<td>-1</td>
</tr>
</tbody>
</table>

The range is $y \leq 0$. The graph of $y = -0.5\sqrt{x}$ is a vertical shrink (by a factor of 0.5) with a reflection in the $x$-axis of the graph of $y = \sqrt{x}$.

**Graphs of Square Root Functions**  
Examples 1 and 2 illustrate the following:

- When $|a| > 1$, the graph of $y = a\sqrt{x}$ is a vertical stretch of the graph of $y = \sqrt{x}$. When $0 < |a| < 1$, the graph of $y = a\sqrt{x}$ is a vertical shrink of the graph of $y = \sqrt{x}$.
- When $a < 0$, the graph of $y = a\sqrt{x}$ is the reflection in the $x$-axis of the graph of $y = |a|\sqrt{x}$.

**Example 3**  
**Graph a function of the form** $y = \sqrt{x} + k$

Graph the function $y = \sqrt{x} + 2$ and identify its domain and range. Compare the graph with the graph of $y = \sqrt{x}$.

**Solution**

To graph the function, make a table, then plot and connect the points. The domain is $x \geq 0$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>2</td>
<td>3</td>
<td>3.4</td>
<td>3.7</td>
<td>4</td>
</tr>
</tbody>
</table>

The range is $y \geq 2$. The graph of $y = \sqrt{x} + 2$ is a vertical translation (of 2 units up) of the graph of $y = \sqrt{x}$.

**Guided Practice** for Examples 1, 2, and 3

Graph the function and identify its domain and range. Compare the graph with the graph of $y = \sqrt{x}$.

1. $y = 2\sqrt{x}$
2. $y = -2\sqrt{x}$
3. $y = \sqrt{x} - 1$
4. $y = \sqrt{x} + 3$
**Example 4**  
Graph a function of the form \( y = \sqrt{x - h} \)

Graph the function \( y = \sqrt{x - 4} \) and identify its domain and range. Compare the graph with the graph of \( y = \sqrt{x} \).

**Solution**
To graph the function, make a table, then plot and connect the points. To find the domain, find the values of \( x \) for which the radicand, \( x - 4 \), is nonnegative. The domain is \( x \geq 4 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>0</td>
<td>1.4</td>
<td>1.7</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

The range is \( y \geq 0 \). The graph of \( y = \sqrt{x - 4} \) is a horizontal translation (of 4 units to the right) of the graph of \( y = \sqrt{x} \).

**Example 5**  
Graph a function of the form \( y = a\sqrt{x - h} + k \)

Graph the function \( y = 2\sqrt{x + 4} - 1 \).

**Step 1** Sketch the graph of \( y = 2\sqrt{x} \). The graph of \( y = 2\sqrt{x} \) starts at the origin and passes through the point \((1, 2)\).

**Step 2** Shift the graph \(|h|\) units horizontally (to the right if \( h \) is positive and to the left if \( h \) is negative) and \(|k|\) units vertically (up if \( k \) is positive and down if \( k \) is negative).

\( h = -4 \) and \( k = -1 \). Shift the graph left 4 units and down 1 unit.

**Guided Practice** for Examples 4 and 5

5. Graph the function \( y = \sqrt{x + 3} \) and identify its domain and range. Compare the graph with the graph of \( y = \sqrt{x} \).

6. Identify the domain and range of the function in Example 5.
EXAMPLE 6  Solve a real-world problem

MICROPHONE SALES  For the period 1988–2002, the amount of sales $y$ (in millions of dollars) of microphones in the United States can be modeled by the function $y = 93\sqrt{x} + 2.2$ where $x$ is the number of years since 1988. Graph the function on a graphing calculator. In what year were microphone sales about $325$ million?

Solution

The graph of the function is shown. Using the trace feature, you can see that $y \approx 325$ when $x \approx 10$. So, microphone sales were about $325$ million 10 years after 1988, or in 1998.

ANOTHER WAY

You can graph $y = 93\sqrt{x} + 2.2$ and $y = 325$. The $x$-coordinate of the point where the graphs intersect represents the year in which sales were about $325$ million.

Guided Practice for Example 6

7. MICROPHONE SALES  Use the function in Example 6 to find the year in which microphone sales were about $250$ million.

11.1 EXERCISES

1. VOCABULARY  Copy and complete: A function containing a radical expression with the independent variable in the radicand is called a(n)  ?  .

2. ★ WRITING  Is the graph of $y = 1.25\sqrt{x}$ a vertical stretch or a vertical shrink of the graph of $y = \sqrt{x}$? Explain your answer.

GRAPHING FUNCTIONS  Graph the function and identify its domain and range. Compare the graph with the graph of $y = \sqrt{x}$.

3. $y = 4\sqrt{x}$
4. $y = 5\sqrt{x}$
5. $y = 0.5\sqrt{x}$
6. $y = 0.25\sqrt{x}$
7. $y = \frac{3}{2} \sqrt{x}$
8. $y = \frac{1}{3} \sqrt{x}$
9. $y = -3\sqrt{x}$
10. $y = -6\sqrt{x}$
11. $y = -0.8\sqrt{x}$
12. $y = -0.75\sqrt{x}$
13. $y = -\frac{1}{4} \sqrt{x}$
14. $y = -\frac{5}{2} \sqrt{x}$
15. ★ MULTIPLE CHOICE  The graph of which function is a vertical shrink of the graph of $y = \sqrt{x}$?
   (A) $y = -5\sqrt{x}$  (B) $y = -\sqrt{x}$  (C) $y = \frac{1}{2} \sqrt{x}$  (D) $y = 8\sqrt{x}$

16. ★ WRITING  The range of the function $y = a\sqrt{x}$ is $y \leq 0$. What can you conclude about the value of $a$? How do you know?
EXAMPLES 3 and 4 on pp. 711–712 for Exs. 17–29

**GRAPHING FUNCTIONS** Graph the function and identify its domain and range. Compare the graph with the graph of \( y = \sqrt{x} \).

17. \( y = \sqrt{x} + 1 \)  
18. \( y = \sqrt{x} + 5 \)  
19. \( y = \sqrt{x} - 3 \)  
20. \( y = \sqrt{x} - 4 \)  
21. \( y = \sqrt{x} + \frac{3}{4} \)  
22. \( y = \sqrt{x} - 4.5 \)  
23. \( y = \sqrt{x + 1} \)  
24. \( y = \sqrt{x} - 6 \)  
25. \( y = \sqrt{x} + 2 \)  
26. \( y = \sqrt{x} + 4 \)  
27. \( y = \sqrt{x} + 1.5 \)  
28. \( y = \sqrt{x - \frac{1}{2}} \)  

29. ★ Multiple Choice The graph of which function is a horizontal translation of 3 units to the right of the graph of \( y = \sqrt{x} \)?

   - A. \( y = \sqrt{x} + 3 \)
   - B. \( y = \sqrt{x} - 3 \)
   - C. \( y = \sqrt{x} + 3 \)
   - D. \( y = \sqrt{x} - 3 \)

**GRAPHING FUNCTIONS** Graph the function.

30. \( y = \sqrt{x + 3} - 2 \)  
31. \( y = \sqrt{x - 2} + 5 \)  
32. \( y = 2\sqrt{x} + 1 \)  
33. \( y = -\sqrt{x + 1} + 2 \)  
34. \( y = -3\sqrt{x} + 2 - 6 \)  
35. \( y = 4\sqrt{x} + 4 - 4 \)  
36. \( y = \frac{1}{2}\sqrt{x - 5} - 3 \)  
37. \( y = -\frac{3}{2}\sqrt{x - 1} - 5 \)  
38. \( y = -\frac{3}{4}\sqrt{x + 8} - 3 \)  

39. ★ Multiple Choice The graph of which function is shown?

   - A. \( y = \sqrt{x + 1} + 2 \)
   - B. \( y = \sqrt{x - 1} + 2 \)
   - C. \( y = \sqrt{x + 1} - 2 \)
   - D. \( y = \sqrt{x - 1} - 2 \)

40. Error Analysis Describe and correct the error in explaining how to graph the function \( y = -5\sqrt{x} - 9 - 10 \).

41. ★ Multiple Choice How is the graph of \( g(x) = 4\sqrt{x} - 3 \) related to the graph of \( h(x) = 4\sqrt{x} + 3 \)?

   - A. It is a vertical stretch by a factor of 3 of the graph of \( h \).
   - B. It is a vertical translation of 3 units down of the graph of \( h \).
   - C. It is a vertical translation of 6 units down of the graph of \( h \).
   - D. It is a horizontal translation of 6 units to the left of the graph of \( h \).

42. Challenge Write a rule for a radical function that has a domain of all real numbers greater than or equal to \(-5\) and a range of all real numbers less than or equal to \(-3\).
11.1 Graph Square Root Functions

EXAMPLE 6

You may wish to use a graphing calculator to complete the following Problem Solving exercises.

43. SUSPENSION BRIDGE The time \( t \) (in seconds) it takes an object dropped from a height \( h \) (in feet) to reach the ground is given by the function \( t = \frac{1}{4} \sqrt{h} \).

a. Graph the function and identify its domain and range.

b. The Royal Gorge Bridge in Canyon City, Colorado, is the world’s highest suspension bridge. It takes about 8 seconds for a stone dropped from the bridge to reach the gorge below. About how high is the bridge?

44. OCEANOGRAPHY Ocean waves can be shallow water, intermediate depth, or deep water waves. The speed \( s \) (in meters per second) of a shallow water wave can be modeled by the function \( s = 3.13 \sqrt{d} \) where \( d \) is the depth (in meters) of the water over which the wave is traveling.

a. Graph the function and identify its domain and range.

b. A tsunami is a type of shallow water wave. Suppose a tsunami has a speed of 200 meters per second. Over approximately what depth of water is the tsunami traveling?

45. LONG JUMP A function for the speed at which a long jumper is running before jumping is \( s = 10.9 \sqrt{h} \) where \( s \) and \( h \) are defined in the diagram. Graph the function and identify its domain and range. To the nearest tenth, approximate the maximum height reached when the long jumper’s speed before jumping is 10.25 meters per second.

46. MULTI-STEP PROBLEM The reading age of written materials is the age at which an average person can read and understand the materials. A function that is sometimes used to identify the reading age \( r \) (in years) of written materials is \( r = \sqrt{w} + 8 \) where \( w \) is the average number of words with 3 or more syllables in samples taken from the written materials.

a. Graph the function and identify its domain and range.

b. What is the average number of words with 3 or more syllables in samples taken from material that can be read and understood by a 10-year-old?
47. **BIOLOGY** Biologists studied two types of duck in the northern Great Plains of the United States from 1987 to 1990. The biologists found functions, given below, that model the number \( y \) of breeding pairs of each type of duck in wetlands with area \( x \) (in hectares).

- **Blue-winged teal:** \( y = 0.7\sqrt{x} \)
- **Northern pintail:** \( y = 0.2\sqrt{x} \)

a. Graph the functions in the same coordinate plane. Identify the domain and range of each function.

b. Find the area (to the nearest hectare) for 1 breeding pair of each type of duck.

48. **★ EXTENDED RESPONSE** The amount of mozzarella cheese \( y \) (in pounds per person) consumed in the United States for the period 1980–2001 can be modeled by \( y = 2\sqrt{x} + 1 \) where \( x \) is the number of years since 1980.

a. **Graph** Graph the function and identify its domain and range.

b. **Apply** In what year was the amount of mozzarella cheese consumed equal to 2 pounds per person?

c. **Explain** In what year was the amount of mozzarella cheese consumed per person double the amount consumed per person in 1980? Explain.

49. **CHALLENGE** The flow rate \( r \) (in gallons per minute) of water through a high-pressure water hose is given by \( r = 29.7d^{2/3}p \) where \( d \) is the nozzle diameter (in inches) and \( p \) is the nozzle pressure (in pounds per square inch). For what value of \( d \) would the graph of the function be identical to the graph of the parent square root function? For what values of \( d \) would the graph be a vertical stretch? a vertical shrink?

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**Mixed Review**

Write the prime factorization of the number if it is not a prime number. If the number is prime, write prime. (p. 910)

- 50. 7
- 51. 14
- 52. 18
- 53. 9
- 54. 24
- 55. 13
- 56. 53
- 57. 72

Evaluate the expression. (p. 110)

- 58. \( \sqrt{16} \)
- 59. \( \sqrt{64} \)
- 60. \( \sqrt{144} \)
- 61. \( \sqrt{900} \)

Factor the expression.

- 62. \( x^2 - 31x + 58 \) (p. 583)
- 63. \( 2x^2 - 7x + 6 \) (p. 593)
- 64. \( 7x^2 + 9x + 2 \) (p. 593)
- 65. \( 6x^2 - 11x - 35 \) (p. 593)
- 66. \( 400x^2 - 9y^2 \) (p. 600)
- 67. \( 25x^2 + 20xy + 4y^2 \) (p. 600)
11.1 Graph Square Root Functions

**Question**
How can you use a graphing calculator to graph square root functions?

**Example**
Graph the function \( y = \sqrt{2x + 3} \) and describe its domain and range.

**Step 1** Enter the function
Enter the function into a graphing calculator. Use parentheses around the radicand.

**Step 2** Graph the function
Graph the function. Adjust the viewing window if necessary.

**Step 3** Describe the domain and range
From the graph, you can see that the domain is all real numbers greater than or equal to \(-1.5\), or \( x \geq -1.5 \). The range is all nonnegative numbers, or \( y \geq 0 \).

**Practice**
Graph the function using a graphing calculator. Then describe the domain and range of the function.

1. \( y = \sqrt{4x} \)
2. \( y = \sqrt{9x} \)
3. \( y = \sqrt{7x} \)
4. \( y = -\sqrt{10x} \)
5. \( y = -3\sqrt{x} \)
6. \( y = 1.5\sqrt{3x} \)
7. \( y = 4.4\sqrt{8x} \)
8. \( y = \sqrt{2x + 8} \)
9. \( y = \sqrt{3x + 4} \)
10. \( y = -\sqrt{2x - 5} \)
11. \( y = -\sqrt{4x - 6} \)
12. \( y = \frac{1}{2}\sqrt{6 - 5x} \)

13. **Roller Coaster** If friction is ignored, the velocity \( v \) (in meters per second) of a roller coaster when it reaches the bottom of a hill can be calculated using the formula \( v = \sqrt{19.6h} \) where \( h \) (in meters) is the height of the hill.

a. Graph the function and describe its domain and range.

b. Use the graph to find the height of a hill if the velocity of the roller coaster at the bottom of the hill is 55 meters per second.
11.2 Properties of Radicals

**MATERIALS** • calculator

**QUESTION** How can you simplify products and quotients of square roots?

**EXPLORE** Simplify products and quotients of square roots

**STEP 1 Find products of square roots**
Copy and complete the table without using a calculator. Compare the values in the second and third columns.

<table>
<thead>
<tr>
<th>Values of $a$ and $b$</th>
<th>Value of $\sqrt{a} \cdot \sqrt{b}$</th>
<th>Value of $\sqrt{ab}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a = 4, b = 9$</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>$a = 9, b = 16$</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>$a = 25, b = 4$</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>$a = 16, b = 36$</td>
<td>?</td>
<td>?</td>
</tr>
</tbody>
</table>

**STEP 2 Find products of square roots**
Use a calculator to copy and complete the table. Compare the values in the second and third columns.

<table>
<thead>
<tr>
<th>Values of $a$ and $b$</th>
<th>Value of $\sqrt{a} \cdot \sqrt{b}$</th>
<th>Value of $\sqrt{ab}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a = 2, b = 3$</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>$a = 10, b = 5$</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>$a = 7, b = 11$</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>$a = 13, b = 6$</td>
<td>?</td>
<td>?</td>
</tr>
</tbody>
</table>

**STEP 3 Find quotients of square roots**
Copy and complete the table without using a calculator. Compare the values in the second and third columns.

<table>
<thead>
<tr>
<th>Values of $a$ and $b$</th>
<th>Value of $\frac{\sqrt{a}}{\sqrt{b}}$</th>
<th>Value of $\sqrt{\frac{a}{b}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a = 4, b = 16$</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>$a = 9, b = 25$</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>$a = 36, b = 4$</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>$a = 4, b = 49$</td>
<td>?</td>
<td>?</td>
</tr>
</tbody>
</table>

**STEP 4 Find quotients of square roots**
Use a calculator to copy and complete the table. Compare the values in the second and third columns.

<table>
<thead>
<tr>
<th>Values of $a$ and $b$</th>
<th>Value of $\frac{\sqrt{a}}{\sqrt{b}}$</th>
<th>Value of $\sqrt{\frac{a}{b}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a = 1, b = 2$</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>$a = 3, b = 8$</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>$a = 12, b = 7$</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>$a = 6, b = 11$</td>
<td>?</td>
<td>?</td>
</tr>
</tbody>
</table>

**DRAW CONCLUSIONS** Use your observations to complete these exercises

In Exercises 1 and 2, copy and complete the statement.

1. The product of two square roots is equal to ?.
2. The quotient of two square roots is equal to ?.

3. **REASONING** Do you think that $\sqrt{a} + \sqrt{b} = \sqrt{a + b}$ for any $a \geq 0$ and any $b \geq 0$? Justify your answer.
A radical expression is in **simplest form** if the following conditions are true:

- No perfect square factors other than 1 are in the radicand.
- No fractions are in the radicand.
- No radicals appear in the denominator of a fraction.

You can use the following property to simplify radical expressions.

**KEY CONCEPT**

*Product Property of Radicals*

**Words** The square root of a product equals the product of the square roots of the factors.

**Algebra** \( \sqrt{ab} = \sqrt{a} \cdot \sqrt{b} \)

**Example** \( \sqrt{4x} = \sqrt{4} \cdot \sqrt{x} = 2\sqrt{x} \)

You can also use the fact that \( \sqrt{a^2} = a \), where \( a \geq 0 \), to simplify radical expressions. In this lesson, whenever a variable appears in the radicand assume that it has only nonnegative values.

**EXAMPLE 1** Use the product property of radicals

<table>
<thead>
<tr>
<th>a. ( \sqrt{32} )</th>
<th>( \sqrt{16} \cdot 2 )</th>
<th>Factor using perfect square factor.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( = \sqrt{16} \cdot 2 )</td>
<td>( = 4\sqrt{2} )</td>
<td>Product property of radicals.</td>
</tr>
<tr>
<td>( = 4\sqrt{2} )</td>
<td>( = 4\sqrt{2} )</td>
<td>Simplify.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>b. ( \sqrt{9x^3} )</th>
<th>( \sqrt{9} \cdot x^2 \cdot x )</th>
<th>Factor using perfect square factors.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( = \sqrt{9} \cdot \sqrt{x^2} \cdot \sqrt{x} )</td>
<td>( = 3x\sqrt{x} )</td>
<td>Product property of radicals.</td>
</tr>
<tr>
<td>( = 3x\sqrt{x} )</td>
<td>( = 3x\sqrt{x} )</td>
<td>Simplify.</td>
</tr>
</tbody>
</table>

**GUIDED PRACTICE** for Example 1

1. Simplify (a) \( \sqrt{24} \) and (b) \( \sqrt{25x^2} \).
EXAMPLE 2  Multiply radicals

a. \[ \sqrt{6} \cdot \sqrt{6} = \sqrt{6 \cdot 6} \]
   \[ = \sqrt{36} \]
   \[ = 6 \]
   Product property of radicals
   Multiply.
   Simplify.

b. \[ \sqrt{3x} \cdot 4\sqrt{x} = 4\sqrt{3x \cdot x} \]
   \[ = 4\sqrt{3x^2} \]
   \[ = 4 \cdot \sqrt{3} \cdot \sqrt{x^2} \]
   Product property of radicals
   Multiply.
   Product property of radicals
   Simplify.

c. \[ \sqrt{7xy^2} \cdot 3\sqrt{x} = 3\sqrt{7xy^2 \cdot x} \]
   \[ = 3\sqrt{7x^2y^2} \]
   \[ = 3 \cdot \sqrt{7} \cdot \sqrt{x^2} \cdot \sqrt{y^2} \]
   Product property of radicals
   Multiply.
   Product property of radicals
   Simplify.

KEY CONCEPT  For Your Notebook

Quotient Property of Radicals

Words  The square root of a quotient equals the quotient of the square roots of the numerator and denominator.

Algebra  \[ \frac{\sqrt{a}}{\sqrt{b}} = \frac{\sqrt{a}}{\sqrt{b}} \]  where \( a \geq 0 \) and \( b > 0 \)

Example  \[ \frac{\sqrt{16}}{\sqrt{25}} = \frac{\sqrt{16}}{\sqrt{25}} = \frac{4}{5} \]

EXAMPLE 3  Use the quotient property of radicals

a. \[ \sqrt{\frac{13}{100}} = \frac{\sqrt{13}}{\sqrt{100}} \]
   \[ = \frac{\sqrt{13}}{10} \]
   Quotient property of radicals
   Simplify.

b. \[ \sqrt{\frac{7}{x^2}} = \frac{\sqrt{7}}{\sqrt{x^2}} \]
   \[ = \frac{\sqrt{7}}{x} \]
   Quotient property of radicals
   Simplify.

GUIDED PRACTICE  for Examples 2 and 3

2. Simplify (a) \( \sqrt{2x^3} \cdot \sqrt{x} \) and (b) \( \frac{1}{\sqrt{y^2}} \).
RATIONALIZING THE DENOMINATOR  Example 4 shows how to eliminate a radical from the denominator of a radical expression by multiplying the expression by an appropriate value of 1. The process of eliminating a radical from an expression’s denominator is called rationalizing the denominator.

**Example 4**  Rationalize the denominator

**a.**  \( \frac{5}{\sqrt{7}} = \frac{5}{\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}} \)

Multiply by \( \frac{\sqrt{7}}{\sqrt{7}} \).

Product property of radicals

\( = \frac{5\sqrt{7}}{\sqrt{49}} \)

\( = \frac{5\sqrt{7}}{7} \)

Simplify.

**b.**  \( \frac{\sqrt{2}}{\sqrt{3b}} = \frac{\sqrt{2}}{\sqrt{3b}} \cdot \frac{\sqrt{3b}}{\sqrt{3b}} \)

Multiply by \( \frac{\sqrt{3b}}{\sqrt{3b}} \).

Product property of radicals

\( = \frac{\sqrt{6b}}{\sqrt{9b^2}} \)

Product property of radicals

\( = \frac{\sqrt{6b}}{\sqrt{9} \cdot \sqrt{b^2}} \)

Simplify.

SUMS AND DIFFERENCES  You can use the distributive property to simplify sums and differences of radical expressions when the expressions have the same radicand.

**Example 5**  Add and subtract radicals

**a.**  \( 4\sqrt{10} + \sqrt{13} - 9\sqrt{10} = 4\sqrt{10} - 9\sqrt{10} + \sqrt{13} \)

Commutative property

\( = (4 - 9)\sqrt{10} + \sqrt{13} \)

Distributive property

\( = -5\sqrt{10} + \sqrt{13} \)

Simplify.

**b.**  \( 5\sqrt{3} + \sqrt{48} = 5\sqrt{3} + \sqrt{16 \cdot 3} \)

Factor using perfect square factor.

\( = 5\sqrt{3} + 4\sqrt{3} \)

Product property of radicals

\( = 5\sqrt{3} + 4\sqrt{3} \)

Simplify.

\( = (5 + 4)\sqrt{3} \)

Distributive property

\( = 9\sqrt{3} \)

Simplify.

**Guided Practice**  for Examples 4 and 5

Simplify the expression.

3. \( \frac{1}{\sqrt{3}} \)  4. \( \frac{1}{\sqrt{x}} \)  5. \( \frac{3}{\sqrt{2x}} \)  6. \( 2\sqrt{7} + 3\sqrt{63} \)
Example 6  Multiply radical expressions

a. \(\sqrt{5}(4 - \sqrt{20}) = 4\sqrt{5} - \sqrt{100}\)  
   \(= 4\sqrt{5} - 10\)  
   Simplify.

b. \((\sqrt{7} + \sqrt{2})(\sqrt{7} - 3\sqrt{2})\)  
   \(= (\sqrt{7})^2 + \sqrt{7}(-3\sqrt{2}) + \sqrt{2} \cdot \sqrt{7} + \sqrt{2}(-3\sqrt{2})\)  
   Multiply.  
   \(= 7 - 3\sqrt{14} - 6\)  
   Simplify.

Example 7  Solve a real-world problem

ASTRONOMY The orbital period of a planet is the time that it takes the planet to travel around the sun. You can find the orbital period \(P\) (in Earth years) using the formula \(P = \sqrt{d^3}\) where \(d\) is the average distance (in astronomical units, abbreviated AU) of the planet from the sun.

a. Simplify the formula.

b. Jupiter’s average distance from the sun is shown in the diagram. What is Jupiter’s orbital period?

Solution

a. \(P = \sqrt{d^3}\)  
   Write formula.

   \(= \sqrt{d^2 \cdot d}\)  
   Factor using perfect square factor.

   \(= \sqrt{d^2} \cdot \sqrt{d}\)  
   Product property of radicals

   \(= d\sqrt{d}\)  
   Simplify.

b. Substitute 5.2 for \(d\) in the simplified formula.
   \(P = 5.2\sqrt{5.2}\)

   The orbital period of Jupiter is \(5.2\sqrt{5.2}\), or about 11.9, Earth years.

Guided Practice for Examples 6 and 7

7. Simplify the expression \((4 - \sqrt{5})(1 - \sqrt{5})\).

8. ASTRONOMY Neptune’s average distance from the sun is about 6 times Jupiter’s average distance from the sun. Is the orbital period of Neptune 6 times the orbital period of Jupiter? Explain.
11.2 **EXERCISES**

1. **VOCABULARY** Copy and complete: The process of eliminating a radical from the denominator of a radical expression is called _____.

2. **WRITING** Is the expression $\sqrt{\frac{2x}{9}}$ written in simplest form? Explain why or why not.

**USING PRODUCT AND QUOTIENT PROPERTIES** Simplify the expression.

3. $\sqrt{20}$
4. $\sqrt{48}$
5. $\sqrt{96}$
6. $\sqrt{72}$
7. $\sqrt{125b}$
8. $\sqrt{4x^2}$
9. $\sqrt{81m^3}$
10. $\sqrt{32m^5}$
11. $\sqrt{5} \cdot \sqrt{30}$
12. $\sqrt{50} \cdot \sqrt{18}$
13. $\sqrt{14x} \cdot \sqrt{2x}$
14. $\sqrt{3b^3} \cdot \sqrt{18b}$
15. $2\sqrt{a^4b^5}$
16. $\sqrt{64x^4r^3}$
17. $\sqrt{m^2n} \cdot \sqrt{n}$
18. $\sqrt{75x^4y} \cdot \sqrt{2x^3}$
19. $\sqrt{\frac{4}{49}}$
20. $\sqrt{\frac{7}{81}}$
21. $\sqrt{\frac{a^3}{121}}$
22. $\sqrt{\frac{100}{4x^2}}$

23. **MULTIPLE CHOICE** Which expression is equivalent to $\sqrt{\frac{9x^2}{16}}$?

- **A** $\sqrt{\frac{3x}{4}}$
- **B** $\frac{3\sqrt{x}}{4}$
- **C** $\frac{3\sqrt{x}}{16}$
- **D** $\frac{3x}{4}$

24. **ERROR ANALYSIS** Describe and correct the error in simplifying the expression $\sqrt{72}$.

$$\sqrt{72} = \sqrt{4 \cdot 18} = 2\sqrt{18}$$

25. **WRITING** Describe two different sequences of steps you could take to simplify the expression $\sqrt{45} \cdot \sqrt{5}$.

**RATIONALIZING THE DENOMINATOR** Simplify the expression.

26. $\frac{2}{\sqrt{2}}$
27. $\frac{4}{\sqrt{3}}$
28. $\sqrt{\frac{5}{48}}$
29. $\sqrt{\frac{4}{52}}$
30. $\frac{3}{\sqrt{a}}$
31. $\frac{1}{\sqrt{2x}}$
32. $\frac{2x^2}{\sqrt{5}}$
33. $\sqrt{\frac{8}{3n^2}}$

**PERFORMING OPERATIONS ON RADICALS** Simplify the expression.

34. $2\sqrt{2} + 6\sqrt{2}$
35. $\sqrt{5} - 6\sqrt{5}$
36. $2\sqrt{6} - 5\sqrt{54}$
37. $9\sqrt{32} + \sqrt{2}$
38. $\sqrt{12} + 6\sqrt{3} + 2\sqrt{6}$
39. $3\sqrt{7} - 5\sqrt{14} + 2\sqrt{28}$
40. $\sqrt{5}(5 - \sqrt{5})$
41. $\sqrt{6}(7\sqrt{3} + 6)$
42. $\sqrt{3}(6\sqrt{2} - 4\sqrt{3})$
43. $(4 - \sqrt{2})(5 + \sqrt{2})$
44. $(2\sqrt{5} + 7)^2$
45. $(\sqrt{7} + \sqrt{3})(6 + \sqrt{8})$
SIMPLIFYING RADICAL EXPRESSIONS Simplify the expression.

46. \(\sqrt{75m^2np^3}\)
47. \(\sqrt{512rs^6} \cdot \sqrt{r^3}\)
48. \(\sqrt{600a\over 4b^3}\)
49. \(\sqrt{50gh^2 \over 125f^3}\)
50. \(\frac{4}{\sqrt{3}} + \frac{7}{\sqrt{12}}\)
51. \(\frac{2\sqrt{6}}{\sqrt{30}} - \frac{3}{\sqrt{20}}\)
52. \(\frac{7}{\sqrt{x}} + \frac{3}{2\sqrt{x}}\)
53. \(\frac{3}{\sqrt{x^3}} + \frac{4}{\sqrt{x}}\)
54. \(\frac{6n}{\sqrt{m^3}} - \frac{8}{\sqrt{m}}\)

CONJUGATES In Exercises 55–62, use the example to simplify the expression.

**Example** Rationalize the denominator using conjugates

Simplify \(\frac{9}{2 - \sqrt{3}}\).

Solution

The binomials \(a\sqrt{b} + c\sqrt{d}\) and \(a\sqrt{b} - c\sqrt{d}\) are called **conjugates**. They differ only by the sign of one term. The product of two conjugates \(a\sqrt{b} + c\sqrt{d}\) and \(a\sqrt{b} - c\sqrt{d}\) does not contain a radical:

\[
(2 + \sqrt{3})(2 - \sqrt{3}) = 2^2 - (\sqrt{3})^2 = 4 - 3 = 1.
\]

You can use conjugates to simplify the expression.

\[
\frac{9}{2 - \sqrt{3}} = \frac{9}{2 - \sqrt{3}} \cdot \frac{2 + \sqrt{3}}{2 + \sqrt{3}} = \frac{9(2 + \sqrt{3})}{(2 - \sqrt{3})(2 + \sqrt{3})} = \frac{18 + 9\sqrt{3}}{4 - 3} = 18 + 9\sqrt{3}
\]

\[
\frac{1}{\sqrt{7} + 1} \quad \frac{2}{5 - \sqrt{3}} \quad \frac{\sqrt{10}}{7 - \sqrt{2}} \quad \frac{\sqrt{5}}{6 + \sqrt{5}}
\]

55. \(\frac{3}{\sqrt{7} + \sqrt{6}}\)
56. \(\frac{11}{\sqrt{11} - \sqrt{7}}\)
57. \(\frac{\sqrt{6}}{\sqrt{2} - \sqrt{3}}\)
58. \(\frac{\sqrt{7} + 1}{\sqrt{7} + \sqrt{2}}\)

63. **REASONING** Multiply the binomials \(a\sqrt{b} + c\sqrt{d}\) and \(a\sqrt{b} - c\sqrt{d}\) to show that the product does not contain a radical.

64. **WRITING** According to the definition of square root, a number \(b\) is a square root of a number \(a\) if \(b^2 = a\). How can you use the definition to show that \(\sqrt{x^2} = x\)? Explain.

65. **MULTIPLYING FUNCTIONS** Let \(f(x) = \sqrt{x} - \sqrt{4x}\), and let \(g(x) = \sqrt{x}\). Find \(h(x) = f(x) \cdot g(x)\).

66. **CHALLENGE** Consider the expression \(\sqrt{2^m}\). Assume \(m\) is a positive integer.

For what values of \(m\) will the expression contain a radical when simplified?

For what values of \(m\) will the expression contain no radical when simplified?

Answer.
67. **FINANCE** You invest $225 in a savings account for two years. The account has an annual interest rate that changes from year to year. You can find the average annual interest rate \( r \) that the account earned over two years using the formula 

\[
r = \sqrt[2]{\frac{V_2}{V_0}} - 1
\]

where \( V_0 \) is the initial investment and \( V_2 \) is the amount in the account after two years. At the end of two years, you have $270 in the account. What was the average annual interest rate (written as a percent) the account earned over two years?

68. **DISTANCE TO THE HORIZON** The distance \( d \) (in miles) that a person can see to the horizon is given by the formula 

\[
d = \sqrt{\frac{3h}{2}}
\]

where \( h \) is the person’s eye level (in feet) above the water. To the nearest mile, find the distance that the person shown can see to the horizon.

69. **MULTI-STEP PROBLEM** You are making a cube-shaped footrest. You want to cover the footrest with fabric. At a fabric store, you choose fabric that costs $6 per square yard.

a. You have $30 to spend on fabric. How much fabric can you buy?

b. The edge length \( s \) (in yards) of the largest footrest you can cover can be found using the formula 

\[
s = \sqrt[3]{\frac{S}{6}}
\]

where \( S \) is the surface area of the footrest (in square yards). Use unit analysis to check the units in the formula.

c. Find the edge length of the largest footrest you can cover to the nearest tenth of a yard.

70. **MULTIPLE REPRESENTATIONS** The velocity \( v \) (in feet per second) of an object that has been dropped can be found using the equation 

\[
v = \sqrt{64d}
\]

where \( d \) is the distance the object falls (in feet) before hitting the ground.

a. **Writing an Equation** Write the equation in simplified form.

b. **Drawing a Graph** Graph the equation. For what value of \( d \) is the velocity about 16 feet per second?

c. **Solving an Equation** Use the equation from part (a) to find the exact value of \( d \) when the velocity is 16 feet per second.
71. ★ SHORT RESPONSE  Physicians can calculate the body surface area $S$ (in square meters) of an adult using the formula $S = \sqrt{\frac{hw}{3600}}$ where $h$ is the adult’s height (in centimeters) and $w$ is the adult’s mass (in kilograms).

a. Simplify the formula.

b. Does an adult who is 1.7 meters tall and has a mass of 70 kilograms have a greater body surface area than an adult who is 1.5 meters tall and has a mass of 70 kilograms? Explain what effect height has on surface area if two people have the same mass.

72. CHALLENGE  The speed $s$ (in miles per hour) at which a vehicle is traveling before an accident is given by $s = \sqrt{30df}$ where $d$ is the length of the skid mark (in feet) and $f$ is the coefficient of friction. The coefficient of friction varies depending on the type of road surface and on the road conditions.

a. A driver is traveling on a newly paved road with a coefficient of friction of 0.80. The driver sees a hazard in the road and is forced to brake. The car skids to a halt leaving a skid mark that is 100 feet long. At what speed was the car traveling when the driver applied the brakes?

b. A perception-reaction time is the amount of time it takes for a person to react to a situation after perceiving it, such as applying the brakes after seeing a hazard in the road. The driver in part (a) has a perception-reaction time of 1.5 seconds. How many feet does the car travel before the driver applies the brakes? Explain how you found your answer.

c. What is the total distance (in feet) traveled from the time the driver in part (a) sees the hazard until the time the car skids to a halt?

Mixed Review

Graph the function and identify its domain and range.

73. $y = -6$ for $x \geq 0$  (p. 215)  
74. $y = -\frac{1}{3}x$  (p. 215)  
75. $5x + y = 3$  (p. 244)

76. $y = 0.4x - 7$  (p. 244)  
77. $y = 4^x$  (p. 520)  
78. $y = (0.4)^x$  (p. 531)

79. $y = -7x^2$  (p. 628)  
80. $y = 5x^2 + 2$  (p. 628)  
81. $y = 3x^2 - 10x - 8$  (p. 635)

82. Write the equation of the line that passes through $(1, 2)$ and is perpendicular to the line $y = \frac{1}{2}x + 3$.  (p. 319)

Solve the equation.  (pp. 575, 583, 593, and 600)

83. $(a - 3)(a + 8) = 0$  
84. $(2y + 3)(2y + 5) = 0$  
85. $-12m^2 + 48 = 0$

86. $x^2 - 8x + 16 = 0$  
87. $d^2 + 8d = -2$  
88. $x^2 - 2x = 24$

89. $10z - 21 = -2z^2$  
90. $5x^2 + 13x + 6 = 0$  
91. $4x^2 - 11x - 3 = 0$

92. $8x^2 - 64 = 0$  
93. $4z^2 = 36$  
94. $b^2 + 12b + 36 = 0$
Solve quadratic equations and check solutions.

In Lesson 10.6, you learned how to find solutions of quadratic equations using the quadratic formula. You can use the method of completing the square and the quotient property of radicals to derive the quadratic formula.

\[
ax^2 + bx + c = 0 \quad \text{Write standard form of a quadratic equation.}
\]

\[
ax^2 + bx = -c \quad \text{Subtract } c \text{ from each side.}
\]

\[
x^2 + \frac{b}{a}x = -\frac{c}{a} \quad \text{Divide each side by } a, a \neq 0.
\]

\[
x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \quad \text{Add } \left(\frac{b}{2a}\right)^2 \text{ to each side to complete the square.}
\]

\[
\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2} \quad \text{Write left side as the square of a binomial.}
\]

\[
\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \quad \text{Simplify right side.}
\]

\[
x + \frac{b}{2a} = \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} \quad \text{Take square roots of each side.}
\]

\[
x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a} \quad \text{Quotient property of radicals}
\]

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{Subtract } \frac{b}{2a} \text{ from each side.}
\]

**SOLVING QUADRATIC EQUATIONS** You can use the quadratic formula and properties of radicals to solve quadratic equations.

**Example 1** Solve an equation

**Solve** \(x^2 - 6x + 3 = 0\).

**Solution**

\[
x^2 - 6x + 3 = 0
\]

Identify \(a = 1\), \(b = -6\), and \(c = 3\).

\[
x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(3)}}{2(1)} \quad \text{Substitute values in the quadratic formula.}
\]

\[
= \frac{6 \pm \sqrt{24}}{2} \quad \text{Simplify.}
\]

\[
= \frac{6 \pm 2\sqrt{6}}{2} \quad \text{Product property of radicals}
\]

\[
= 3 \pm \sqrt{6} \quad \text{Simplify.}
\]

The solutions of the equation are \(3 + \sqrt{6}\) and \(3 - \sqrt{6}\).
EXAMPLE 2  Check the solutions of an equation

Check the solutions of the equation from Example 1.

Solution
The solutions of \( x^2 - 6x + 3 = 0 \) are \( 3 + \sqrt{6} \) and \( 3 - \sqrt{6} \). You can check each solution by substituting it into the original equation.

Check \( x = 3 + \sqrt{6} \):

\[
\begin{align*}
& x^2 - 6x + 3 = 0 \\
& (3 + \sqrt{6})^2 - 6(3 + \sqrt{6}) + 3 = 0 \\
& 9 + 6\sqrt{6} + 6 - 18 - 6\sqrt{6} + 3 = 0 \\
& 0 = 0 \checkmark \text{ Solution checks.}
\end{align*}
\]

Check \( x = 3 - \sqrt{6} \):

\[
\begin{align*}
& x^2 - 6x + 3 = 0 \\
& (3 - \sqrt{6})^2 - 6(3 - \sqrt{6}) + 3 = 0 \\
& 9 - 6\sqrt{6} + 6 - 18 + 6\sqrt{6} + 3 = 0 \\
& 0 = 0 \checkmark \text{ Solution checks.}
\end{align*}
\]

PRACTICE

Solve the equation using the quadratic formula. Check the solution.

1. \( x^2 + 4x + 2 = 0 \)
2. \( x^2 + 6x - 1 = 0 \)
3. \( x^2 + 8x + 8 = 0 \)
4. \( x^2 - 7x + 1 = 0 \)
5. \( 3x^2 + 6x - 1 = 0 \)
6. \( 2x^2 - 4x - 3 = 0 \)
7. \( 5x^2 - 2x - 2 = 0 \)
8. \( 4x^2 + 10x + 3 = 0 \)
9. \( x^2 - x - 3 = 0 \)
10. \( x^2 - 2x - 8 = 0 \)
11. \( -x^2 + 7x + 3 = 0 \)
12. \( x^2 + 3x - 9 = 0 \)
13. \( \frac{-5}{2}x^2 + 10x - 5 = 0 \)
14. \( \frac{1}{2}x^2 + 3x - 9 = 0 \)
15. \( 3x^2 - 2 = 0 \)
16. \( -2x^2 - 7x = 0 \)
17. \( 3x^2 + x = 6 \)
18. \( x^2 - 4x = -2 \)
19. Show that \( \frac{-b + \sqrt{b^2 - 4ac}}{2a} \) and \( \frac{-b - \sqrt{b^2 - 4ac}}{2a} \) are solutions of \( ax^2 + bx + c = 0 \) by substituting.
20. Derive a formula to find solutions of equations that have the form \( ax^2 + x + c = 0 \). Use your formula to find solutions of \( -2x^2 + x + 8 = 0 \).
21. Find the sum and product of \( \frac{-b + \sqrt{b^2 - 4ac}}{2a} \) and \( \frac{-b - \sqrt{b^2 - 4ac}}{2a} \). Write a quadratic expression whose solutions have a sum of 2 and a product of \( \frac{1}{2} \).
22. What values can \( a \) have in the equation \( ax^2 + 12x + 3 = 0 \) in order for the equation to have one or two real solutions? Explain.
11.3 Solve Radical Equations

**Before**
You solved linear, quadratic, and exponential equations.

**Now**
You will solve radical equations.

**Why?**
So you can use scientific formulas to study animals, as in Ex. 39.

**Key Vocabulary**
- radical equation
- extraneous solution

An equation that contains a radical expression with a variable in the radicand is a **radical equation**. To solve a radical equation, you need to isolate the radical on one side and then square both sides of the equation.

**KEY CONCEPT**

**For Your Notebook**

**Squaring Both Sides of an Equation**

**Words** If two expressions are equal, then their squares are equal.

**Algebra** If \( a = b \), then \( a^2 = b^2 \).

**Example** If \( \sqrt{x} = 3 \), then \( (\sqrt{x})^2 = 3^2 \).

**Example 1** Solve a radical equation

Solve \( 2\sqrt{x} - 8 = 0 \).

**Solution**

\[
\begin{align*}
2\sqrt{x} - 8 &= 0 & \text{Write original equation.} \\
2\sqrt{x} &= 8 & \text{Add 8 to each side.} \\
\sqrt{x} &= 4 & \text{Divide each side by 2.} \\
(\sqrt{x})^2 &= 4^2 & \text{Square each side.} \\
x &= 16 & \text{Simplify.}
\end{align*}
\]

The solution is 16.

**CHECK** Check the solution by substituting it in the original equation.

\[
\begin{align*}
2\sqrt{16} - 8 &= 0 & \text{Write original equation.} \\
2\sqrt{16} - 8 &= 0 & \text{Substitute 16 for } x. \\
2 \cdot 4 - 8 &= 0 & \text{Simplify.} \\
0 &= 0 \checkmark & \text{Solution checks.}
\end{align*}
\]

**Guided Practice** for Example 1

1. Solve (a) \( \sqrt{x} - 7 = 0 \) and (b) \( 12\sqrt{x} - 3 = 0 \).
**Example 2** Solve a radical equation

Solve $4\sqrt{x - 7} + 12 = 28$.

**Solution**

\[
\begin{align*}
4\sqrt{x - 7} + 12 &= 28 & \text{Write original equation.} \\
4\sqrt{x - 7} &= 16 & \text{Subtract 12 from each side.} \\
\sqrt{x - 7} &= 4 & \text{Divide each side by 4.} \\
(\sqrt{x - 7})^2 &= 4^2 & \text{Square each side.} \\
x - 7 &= 16 & \text{Simplify.} \\
x &= 23 & \text{Add 7 to each side.}
\end{align*}
\]

The solution is 23.

**CHECK** To check the solution using a graphing calculator, first rewrite the equation so that one side is 0: $4\sqrt{x - 7} - 16 = 0$. Then graph the related equation $y = 4\sqrt{x - 7} - 16$. You can see that the graph crosses the $x$-axis at $x = 23$.

**Example 3** Solve an equation with radicals on both sides

Solve $\sqrt{3x - 17} = \sqrt{x + 21}$.

**Solution**

\[
\begin{align*}
\sqrt{3x - 17} &= \sqrt{x + 21} & \text{Write original equation.} \\
(\sqrt{3x - 17})^2 &= (\sqrt{x + 21})^2 & \text{Square each side.} \\
3x - 17 &= x + 21 & \text{Simplify.} \\
2x - 17 &= 21 & \text{Subtract } x \text{ from each side.} \\
2x &= 38 & \text{Add 17 to each side.} \\
x &= 19 & \text{Divide each side by 2.}
\end{align*}
\]

The solution is 19. Check the solution.

**Guided Practice** for Examples 2 and 3

Solve the equation.

2. $\sqrt{x - 5} + 7 = 12$  
3. $\sqrt{x} + 4 = \sqrt{2x - 1}$  
4. $\sqrt{4x - 3} - \sqrt{x} = 0$

**Extraneous Solutions** Squaring both sides of the equation $a = b$ can result in a solution of $a^2 = b^2$ that is not a solution of the original equation. Such a solution is called an extraneous solution. When you square both sides of an equation, check each solution in the original equation to be sure there are no extraneous solutions.
Example 4  Solve an equation with an extraneous solution

Solve \( \sqrt{6 - x} = x \).

- **Write original equation.**
- **Square each side.**
- **Simplify.**
- **Write in standard form.**
- **Factor.**
- **Zero-product property**
- **Solve for \( x \).**

**CHECK** Check 2 and \(-3\) in the original equation.

- If \( x = 2 \): \( \sqrt{6 - 2} \neq 2 \)
- If \( x = -3 \): \( \sqrt{6 - (-3)} \neq -3 \)

Because \(-3\) does not check in the original equation, it is an extraneous solution. The only solution of the equation is 2.

Example 5  Solve a real-world problem

**SAILING** The hull speed \( s \) (in nautical miles per hour) of a sailboat can be estimated using the formula \( s = 1.34 \sqrt{l} \) where \( l \) is the length (in feet) of the sailboat’s waterline, as shown. Find the length (to the nearest foot) of the sailboat’s waterline if it has a hull speed of 8 nautical miles per hour.

**Solution**

- **Write original equation.**
- **Substitute 8 for \( s \).**
- **Divide each side by 1.34.**
- **Square each side.**
- **Simplify.**

The sailboat has a waterline length of about 36 feet.

Guided Practice  for Examples 4 and 5

5. Solve \( \sqrt{3}x + 4 = x \).

6. **WHAT IF?** In Example 5, suppose the sailboat’s hull speed is 6.5 nautical miles per hour. Find the sailboat’s waterline length to the nearest foot.
1. **VOCABULARY** Copy and complete: To find the solution of $\sqrt{12} - x = x$, you square both sides of the equation and solve. The solutions of $(\sqrt{12} - x)^2 = x^2$ are $-4$ and $3$, but $-4$ is a(n) __ of $\sqrt{12} - x = x$.

2. **WRITING** Is $x + x\sqrt{2} = 4$ a radical equation? Explain why or why not.

### SOLVING EQUATIONS
Solve the equation. Check for extraneous solutions.

3. $3\sqrt{x} - 6 = 0$
4. $2\sqrt{x} - 9 = 0$
5. $\sqrt{3x} + 4 = 16$
6. $\sqrt{5x} + 5 = 0$
7. $\sqrt{x} + 7 + 5 = 11$
8. $\sqrt{x} - 8 - 4 = -2$
9. $2\sqrt{x} - 4 - 2 = 2$
10. $3\sqrt{x} - 1 - 5 = 5$
11. $\sqrt{6 - 2x} + 12 = 21$
12. $5\sqrt{x} - 3 + 4 = 14$
13. $2\sqrt{x} - 11 - 8 = 4$
14. $\sqrt{3x} - 2 = \sqrt{x}$
15. $\sqrt{x} - 2x = \sqrt{x} - x$
16. $\sqrt{3x} + 8 = \sqrt{x} + 4$
17. $9x - 30 = \sqrt{4x} + 5$
18. $\sqrt{21} - x - \sqrt{1} - x = 0$
19. $\sqrt{x} - 12 - \sqrt{x} - 8 = 0$
20. $\sqrt{\frac{1}{2} x - 2} - \sqrt{x} - 8 = 0$

21. **MULTIPLE CHOICE** Which is the solution of the equation $10\sqrt{x} + 3 + 3 = 18$?
   - **A** $\frac{3}{2}$
   - **B** $\frac{3}{4}$
   - **C** $\frac{3}{4}$
   - **D** $\frac{3}{2}$

### SOLVING EQUATIONS
Solve the equation. Check for extraneous solutions.

22. $x = \sqrt{42 - x}$
23. $\sqrt{4} - 3x = x$
24. $\sqrt{11x} - 24 = x$
25. $\sqrt{14x} - 3 = 4x$
26. $2x = \sqrt{1} - 3x$
27. $\sqrt{2} - x = x + 4$

### ERROR ANALYSIS
Describe and correct the error in solving the equation.

28. $\sqrt{3x + 9} = 0$
   - $\sqrt{3x} = -9$
   - $3x = 81$
   - $x = 27$

29. $x = \sqrt{18 - 7x}$
   - $x^2 = 18 - 7x$
   - $x^2 + 7x - 18 = 0$
   - $(x - 2)(x + 9) = 0$
   - $x = 2$ or $x = -9$

### GEOMETRY
The formula for the slant height $s$ (in inches) of a cone is $s = \sqrt{h^2 + r^2}$ where $h$ is the height of the cone (in inches) and $r$ is the radius of its base (in inches), as shown. Find the height of the cone if you know the slant height is 4 inches and the radius is 2 inches.
**SOLVING EQUATIONS** Solve the equation. Check for extraneous solutions.

31. $\sqrt{x} + 2 = \sqrt{x} - 1$
32. $2 - \sqrt{x} + 1 = \sqrt{x} + 3$
33. $\sqrt{5x} + 9 + \sqrt{5x} = 9$

34. ★ WRITING A student solves the equation $\sqrt{x} + 2 = x$ and finds that $x = 2$ or $x = -1$. Without checking by substituting into the equation, which is the extraneous solution, 2 or -1? How do you know?

35. CHALLENGE Write a radical equation that has 3 and 4 as solutions.

**PROBLEM SOLVING**

36. FORESTS The dark green areas on the image shown represent regions with heavy foliage. In Texas, the area of land $y$ (in millions of acres) that was covered by forest during the period 1907–2002 can be modeled by the function $y = 2.5\sqrt{143} - x$ where $x$ is the number of years since 1907. In what year were about 20 million acres of land covered by forest in Texas?

37. PER CAPITA CONSUMPTION The annual banana consumption $y$ (in pounds per person) in the United States for the period 1970–2000 can be modeled by the function $y = \sqrt{18x} + 272$ where $x$ is the number of years since 1970. In what year were about 20 pounds of bananas consumed per person?

38. MULTI-STEP PROBLEM The velocity $v$ (in meters per second) at which a trapeze performer swings can be modeled by the function $v = \sqrt{19.6d}$ where $d$ is the difference (in meters) between the highest and lowest position of the performer’s center of gravity during the swing.
   a. A trapeze performer swings at a velocity of 5 meters per second. What is the value of $d$?
   b. Suppose the performer jumps straight up off the starting board, increasing the velocity of the swing by 0.4 meter per second. By how many meters does the value of $d$ increase?

39. BIOLOGY A bushbaby is a small animal that can perform standing jumps of over 2 meters. Scientists found that the time $t$ (in seconds) in which a bushbaby must extend its legs in order to jump to a height $h$ (in meters) is given by the function $t = 0.45\ell\frac{1}{\sqrt{h}}$ where $\ell$ is the length of the bushbaby’s legs (in meters). A particular bushbaby has a leg length of 0.16 meter. The bushbaby can extend its legs in 0.05 second. About how high does the bushbaby jump? Round your answer to the nearest tenth of a meter.
40. **SHORT RESPONSE** The amount of time \( t \) (in seconds) it takes a simple pendulum to complete one full swing is called the period of the pendulum and is given by \( t = 2\pi \sqrt{\frac{l}{32}} \) where \( l \) is the length of the pendulum (in feet).

a. **Apply** A visitor at a museum notices that a pendulum on display has a period of about 11 seconds. About how long is the pendulum? Use 3.14 for \( \pi \) and round your answer to the nearest foot.

b. **Explain** Does increasing the length of a pendulum increase or decrease its period? Explain.

41. **CHALLENGE** The frequency \( f \) (in cycles per second) of a string of an electric guitar is given by the equation \( f = \frac{1}{2l} \sqrt{\frac{T}{m}} \) where \( l \) is the length of the string (in meters), \( T \) is the string’s tension (in newtons), and \( m \) is the string’s mass per unit length (in kilograms per meter). The high E string of a particular electric guitar is 0.64 meter long with a mass per unit length of 0.000401 kilogram per meter. How much tension is required to produce a frequency of about 330 cycles per second? Would you need more or less tension if you want to create the same frequency on a string with greater mass per unit length? Explain.

### Mixed Review

Make a scatter plot of the data. **Describe** the correlation of the data. If possible, fit a line to the data and write an equation for the line. *(p. 325)*

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<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
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<table>
<thead>
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</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>0</td>
<td>11</td>
<td>24</td>
<td>31</td>
<td>44</td>
</tr>
</tbody>
</table>

42. Solve the equation.

44. \( x^2 + 21x + 20 = 0 \) *(p. 583)*
45. \( x^2 - 21x + 38 = 0 \) *(p. 583)*
46. \( x^2 - 8x - 9 = 0 \) *(p. 583)*
47. \( 11x^2 - 11 = 0 \) *(p. 600)*
48. \( 5x^2 - 125 = 0 \) *(p. 600)*
49. \( 8x^2 - 32 = 0 \) *(p. 600)*

### Quiz for Lessons 11.1–11.3

1. Graph the function \( y = \sqrt{x - 3} \) and identify its domain and range.
   Compare the graph with the graph of \( y = \sqrt{x} \). *(p. 710)*

2. \( \sqrt{150} \)
3. \( \sqrt{2c^2} \cdot \sqrt{8c} \)
4. \( (7 + \sqrt{5})(2 - \sqrt{5}) \)
5. \( \frac{14}{\sqrt{2}} \)
6. \( \frac{98}{x^6} \)
7. \( \sqrt{\frac{80x^3}{5y}} \)

Solve the equation. Check for extraneous solutions. *(p. 729)*

8. \( \sqrt{x} - 15 = 0 \)
9. \( \sqrt{4x} - 7 = \sqrt{2x} + 19 \)
10. \( \sqrt{6x - 5} = x \)
Lessons 11.1–11.3

1. OPEN-ENDED The velocity $v$ (in meters per second) of a car moving in a circular path that has radius $r$ (in meters) is given by $v = \sqrt{\frac{Fr}{m}}$ where $F$ is the force (in newtons) pulling the car toward the center of the circular path and $m$ is the mass of the car (in kilograms).

A 1200 kilogram car is traveling at a constant velocity of 20 meters per second in a circular path of radius $r$ meters where $r \geq 100$. Choose two different values of $r$ to show how the force, $F$, acting on the car changes as the radius increases.

2. MULTI-STEP PROBLEM The number $y$ of companies listed on the New York Stock Exchange for the period 1999–2002 can be modeled by the function $y = 3018 - 146\sqrt{x}$ where $x$ is the number of years since 1999.

a. Graph the function.

b. How many companies were listed on the New York Stock Exchange in 1999?

c. In what year were there about 220 companies fewer than the number of companies listed on the New York Stock Exchange in 1999?

3. GRIDDED ANSWER The voltage $V$ (in volts) of an amplifier is given by the function $V = \sqrt{PR}$ where $P$ is the power (in watts) and $R$ is the resistance (in ohms). A particular amplifier produces 10 volts and has a resistance of 4 ohms. How many watts are produced by the amplifier?

4. MULTI-STEP PROBLEM For the period 1994–2001, the annual consumption of corn products $y$ (in pounds per person) in the United States can be modeled by the function $y = 6.1\sqrt{x} + 15.8$ where $x$ is the number of years since 1994.

a. Graph the function.

b. In what year were about 25 pounds of corn products consumed per person?

5. EXTENDED RESPONSE Competitors in a ski mountaineering race must climb a mountain and ski down it as quickly as possible. The race begins with the firing of a starting gun. Near Earth’s surface, the speed of sound $s$ (in meters per second) through air is given by $s = 20\sqrt{T + 273}$ where $T$ is the air temperature (in degrees Celsius).

a. A typical temperature at the start of a race is $-5^\circ$C. What is the speed of sound at this temperature?

b. The person firing the starting gun is standing 50 meters away from the racers. How long will it take for the racers to hear the starting gun?

c. What happens to the time it takes for the racers to hear the starting gun if the temperature at the start of the race is lower? Explain.

6. SHORT RESPONSE A person’s maximum running speed $s$ (in meters per second) can be approximated by the function $s = \pi \sqrt{\frac{9.8l}{6}}$ where $l$ is the person’s leg length (in meters).

a. To the nearest tenth of a meter, what is the leg length of a person whose maximum running speed is about 3.8 meters per second?

b. What happens to running speed as leg length increases? Explain.
11.4 The Pythagorean Theorem

MATERIALS • graph paper • scissors

QUESTION How are the lengths of the sides of a right triangle related to each other?

EXPLORE Examine the relationship among the lengths of the sides of a right triangle

STEP 1 Make right triangles
Cut a right triangle out of graph paper. Make three copies of it.

STEP 2 Arrange as a square
Arrange the right triangles to form a square within a square, as shown.

DRAW CONCLUSIONS Use your observations to complete these exercises

1. How are the areas of the triangles and inner square related to the area of the outer square?

In Exercises 2–4, let \(a\), \(b\), and \(c\) be the lengths of the sides of a right triangle with \(a < b < c\), as shown. Write an expression for the area of the figure described below.

2. One of the right triangles in terms of \(a\) and \(b\)
3. The outer square in terms of \(c\)
4. The inner square in terms of \(a\) and \(b\)

5. Use the relationship you determined in Exercise 1 and your results from Exercises 2–4 to write an equation that relates \(a\), \(b\), and \(c\). Simplify the equation.

6. REASONING The triangle shown is a right triangle. Find the value of \(x\). Explain how you found your answer.
11.4 Apply the Pythagorean Theorem and Its Converse

You solved radical equations.

You will use the Pythagorean theorem and its converse.

So you can examine angles in architecture, as in Ex. 35.

Key Vocabulary
- hypotenuse
- legs of a right triangle
- Pythagorean theorem

The hypotenuse of a right triangle is the side opposite the right angle. It is the longest side of a right triangle. The legs are the two sides that form the right angle.

A theorem is a statement that can be proved true. The Pythagorean theorem states the relationship among the lengths of the sides of a right triangle.

**KEY CONCEPT**

For Your Notebook

The Pythagorean Theorem

Words If a triangle is a right triangle, then the sum of the squares of the lengths of the legs equals the square of the length of the hypotenuse.

Algebra \[ a^2 + b^2 = c^2 \]

**EXAMPLE 1** Use the Pythagorean theorem

Find the unknown length for the triangle shown.

Solution

\[ a^2 + b^2 = c^2 \]  \hspace{1cm} \text{Pythagorean theorem}
\[ a^2 + 6^2 = 7^2 \]  \hspace{1cm} \text{Substitute 6 for } b \text{ and 7 for } c.
\[ a^2 + 36 = 49 \]  \hspace{1cm} \text{Simplify.}
\[ a^2 = 13 \]  \hspace{1cm} \text{Subtract 36 from each side.}
\[ a = \sqrt{13} \]  \hspace{1cm} \text{Take positive square root of each side.}

The side length \( a \) is \( \sqrt{13} \).

**GUIDED PRACTICE** for Example 1

1. The lengths of the legs of a right triangle are \( a = 5 \) and \( b = 12 \). Find \( c \).
EXAMPLE 2

Use the Pythagorean theorem

A right triangle has one leg that is 2 inches longer than the other leg. The length of the hypotenuse is $\sqrt{10}$ inches. Find the unknown lengths.

Solution

Sketch a right triangle and label the sides with their lengths. Let $x$ be the length of the shorter leg.

\[
\begin{align*}
    a^2 + b^2 &= c^2 & \text{Pythagorean theorem} \\
    x^2 + (x + 2)^2 &= (\sqrt{10})^2 & \text{Substitute.} \\
    x^2 + x^2 + 4x + 4 &= 10 & \text{Simplify.} \\
    2x^2 + 4x - 6 &= 0 & \text{Write in standard form.} \\
    2(x - 1)(x + 3) &= 0 & \text{Factor.} \\
    x - 1 &= 0 \text{ or } x + 3 = 0 & \text{Zero-product property} \\
    x &= 1 \text{ or } x = -3 & \text{Solve for } x.
\end{align*}
\]

$\triangleright$ Because length is nonnegative, the solution $x = -3$ does not make sense. The legs have lengths of 1 inch and $1 + 2 = 3$ inches.

EXAMPLE 3

Standardized Test Practice

A soccer player makes a corner kick to another player, as shown. To the nearest yard, how far does the player kick the ball?

A \hspace{1cm} 7 yards  \hspace{1cm} B \hspace{1cm} 38 yards  \hspace{1cm} C \hspace{1cm} 42 yards  \hspace{1cm} D \hspace{1cm} 52 yards

Solution

The path of the kicked ball is the hypotenuse of a right triangle. The length of one leg is 12 yards, and the length of the other leg is 40 yards.

\[
\begin{align*}
    c^2 &= a^2 + b^2 & \text{Pythagorean theorem} \\
    c^2 &= 12^2 + 40^2 & \text{Substitute 12 for } a \text{ and 40 for } b. \\
    c^2 &= 1744 & \text{Simplify.} \\
    c &= \sqrt{1744} \approx 42 & \text{Take positive square root of each side.}
\end{align*}
\]

$\triangleright$ The correct answer is C.  \hspace{1cm} A \hspace{1cm} B \hspace{1cm} C \hspace{1cm} D

Guided Practice for Examples 2 and 3

2. A right triangle has one leg that is 3 inches longer than the other leg. The length of the hypotenuse is 15 inches. Find the unknown lengths.

3. SWIMMING A rectangular pool is 30 feet wide and 60 feet long. You swim diagonally across the pool. To the nearest foot, how far do you swim?
11.4 Apply the Pythagorean Theorem and Its Converse

**EXAMPLE 4** Determine right triangles

Tell whether the triangle with the given side lengths is a right triangle.

a. 8, 15, 17

\[ 8^2 + 15^2 = 17^2 \]

\[ 64 + 225 = 289 \]

\[ 289 = 289 \checkmark \]

\( \rightarrow \) The triangle is a right triangle.

b. 5, 8, 9

\[ 5^2 + 8^2 \neq 9^2 \]

\[ 25 + 64 \neq 81 \]

\[ 89 \neq 81 \times \]

\( \rightarrow \) The triangle is not a right triangle.

**EXAMPLE 5** Use the converse of the Pythagorean theorem

**CONSTRUCTION** A construction worker is making sure one corner of the foundation of a house is a right angle. To do this, the worker makes a mark 8 feet from the corner along one wall and another mark 6 feet from the same corner along the other wall. The worker then measures the distance between the two marks and finds the distance to be 10 feet. Is the corner a right angle?

**Solution**

\[ 8^2 + 6^2 = 10^2 \]

\[ 64 + 36 = 100 \]

\[ 100 = 100 \checkmark \]

\( \rightarrow \) Because the sides that the construction worker measured form a right triangle, the corner of the foundation is a right angle.

**GUIDED PRACTICE** for Examples 4 and 5

Tell whether the triangle with the given side lengths is a right triangle.

4. 7, 11, 13

5. 15, 36, 39

6. 15, 112, 113

7. **WINDOW DESIGN** A window has the shape of a triangle with side lengths of 120 centimeters, 120 centimeters, and 180 centimeters. Is the window a right triangle? *Explain.*
11.4 EXERCISES

1. VOCABULARY Copy and complete: In a right triangle, the side opposite the right angle is called the ___.

2. ★ WRITING Explain how you can tell whether a triangle with side lengths of 9, 12, and 15 is a right triangle.

USING THE PYTHAGOREAN THEOREM Let $a$ and $b$ represent the lengths of the legs of a right triangle, and let $c$ represent the length of the hypotenuse. Find the unknown length.

3. $a = 3, c = 5$  
4. $b = 3, c = 7$  
5. $a = 5, b = 6$

6. $b = 5, c = 10$  
7. $a = 8, b = 8$  
8. $a = 5, b = 12$

9. $a = 8, b = 12$  
10. $a = 7, c = 25$

11. $b = 15, c = 17$

12. $a = 9, c = 41$

13. $b = 3, c = 3.4$

14. $a = 1.2, c = 3.7$

15. ★ MULTIPLE CHOICE A tennis court is 36 feet by 78 feet. What is the length of a diagonal? Round your answer to the nearest tenth of a foot.

   A 42.0 feet  
   B 69.2 feet  
   C 85.9 feet  
   D 114.0 feet

16. ERROR ANALYSIS Describe and correct the error in finding the unknown length.

   \[ 18^2 + 30^2 = x^2 \]
   \[ 1224 = x^2 \]
   \[ 34 = x \]

USING THE PYTHAGOREAN THEOREM Find the unknown lengths.

17. $x + 1$  
18. $2x - 3$  
19. $5x - 1$

20. A right triangle has one leg that is 2 inches longer than the other leg. The length of the hypotenuse is $\sqrt{130}$ inches. Find the lengths of the legs.

21. A right triangle has one leg that is 3 times as long as the other leg. The length of the hypotenuse is $\sqrt{40}$ inches. Find the lengths of the legs.

22. A right triangle has one leg that is $\frac{1}{2}$ of the length of the other leg. The length of the hypotenuse is $6\sqrt{5}$ inches. Find the lengths of the legs.

DETERMINING RIGHT TRIANGLES Tell whether the triangle with the given side lengths is a right triangle.

23. 2, 3, 4  
24. 9, 12, 15  
25. 8, 16, 18

26. 9, 21, 24  
27. 11, 60, 61  
28. 24, 143, 145
29. ★ MULTIPLE CHOICE What is the area of the largest square in the coordinate plane shown?
   A. 100 square units
   B. 64 square units
   C. 36 square units
   D. 25 square units

30. ★ WRITING Given that two side lengths of a right triangle are 11 inches and 6 inches, is it possible to find the length of the third side? Explain.

31. REASONING A Pythagorean triple is a group of integers $a$, $b$, and $c$ that represent the side lengths of a right triangle. For example, the integers 3, 4, and 5 form a Pythagorean triple. Choose any two positive integers $m$ and $n$ such that $m < n$. Then find $a$, $b$, and $c$ as follows: $a = n^2 - m^2$, $b = 2mn$, and $c = n^2 + m^2$. Show that the numbers you generated form a Pythagorean triple. Then use the converse of the Pythagorean theorem to show that the equations for $a$, $b$, and $c$ always generate Pythagorean triples.

32. CHALLENGE The edge length of the cube is 7 inches.
   a. Find the value of $x$.
   b. Find the value of $y$.

PROBLEM SOLVING

33. ARCHITECTURE An earthquake-resistant building has dampers built into its structure to help minimize damage caused by an earthquake. A section of the structural frame of such a building is shown. What is the length of the damper? Round your answer to the nearest foot.

34. SAILS A sail has the shape of a triangle. The side lengths are 146 inches, 131 inches, and 84 inches. Is the sail a right triangle? Explain.
35. **FLATIRON BUILDING** A top view of the Flatiron Building in New York City is shown. The triangle indicates the basic shape of the building's roof. Is the triangle a right triangle? *Explain.*

36. **SCREEN SIZES** The size of a television is indicated by the length of a diagonal of the television screen. The aspect ratio of a television screen is the ratio of the length of the screen to the width of the screen. The size of a particular television is 30 inches, and its aspect ratio is $4 : 3$. What are the width and the length of the television screen?

37. ★ **EXTENDED RESPONSE** The *Wheel of Theodorus* is a figure formed by a chain of right triangles with consecutive triangles sharing a common side. The hypotenuse of one triangle becomes a leg of the next, as shown.

   a. **Calculate** What is the length of the longest hypotenuse in the diagram?

   b. **Extend** Extend the diagram to include two more triangles. What is the length of the longest hypotenuse in the new diagram?

   c. **Analyze** Find a formula for the length of the hypotenuse of the $n$th triangle. *Explain* how you found your answer.

38. **CHALLENGE** A baseball diamond has the shape of a square with side lengths of 90 feet. A catcher wants to get a player running from first base to second base out, so the catcher must throw the ball to second base before the runner reaches second base.

   a. The catcher is 5 feet behind home plate. How far does the catcher have to throw the ball to reach second base? Round your answer to the nearest foot.

   b. The catcher throws the ball at a rate of 90 feet per second when the player is 30 feet away from second base. Will the catcher get the player out if the player is running at a rate of 22 feet per second? *Explain.*

---

**Mixed Review**

Plot the point in a coordinate plane. *Describe* the location of the point. *(p. 206)*

39. $(5, 4)$  
40. $(-2, 6)$  
41. $(-1, -3)$  
42. $(0, 5)$  
43. $(0, 0)$  
44. $(-8, -2)$  
45. $(-1.5, 0)$  
46. $(3.25, -2.5)$

Evaluate the expression.

47. $5x^2$ when $x = 3$ *(p. 8)*  
48. $-|x| - 8$ when $x = -2$ *(p. 64)*  
49. $\sqrt{x - y}$ when $x = 9, y = -7$ *(p. 110)*  
50. $\sqrt{xy}$ when $x = 27, y = 3$ *(p. 110)*
11.5 Distance in The Coordinate Plane

**Materials**
- graph paper

**Question**
How can you find the distance between two points?

**Explore**
Find the distance between points $A(-3, -2)$ and $B(4, -2)$

**Step 1: Plot points**
Plot the points $A(-3, -2)$ and $B(4, -2)$ in the same coordinate plane.

**Step 2: Find distance**
Find the distance between the points by counting the grid lines between them.

**Step 3: Find distance**
Find the distance by subtracting the $x$-coordinate of point $A$ from the $x$-coordinate of point $B$.

**Step 4: Compare results**
How does your result from Step 2 compare with your result from Step 3?

**Draw Conclusions**
Use your observations to complete these exercises

1. Subtract the $x$-coordinate of point $B$ from the $x$-coordinate of point $A$. How is the value different from the values found in Steps 2 and 3 above? How could you make them the same?

2. Assume points $C(x_1, y_1)$ and $D(x_2, y_2)$ lie on the same horizontal line. Write an expression that can be used to find the distance between the points.

3. Assume points $C(x_1, y_1)$ and $D(x_2, y_2)$ lie on the same vertical line. Write an expression that can be used to find the distance between the points. Check your expression using $(-2, 4)$ and $(-2, -3)$.

In Exercises 4–12, find the distance between the two points.

4. $(2, 3), (-5, 3)$
5. $(0, -4), (7, -4)$
6. $(-1, 5), (2, 5)$
7. $(4, -6), (6, -6)$
8. $(-5, -4), (-2, -4)$
9. $(2, 8), (2, 3)$
10. $(5, -6), (5, -2)$
11. $(0, -4), (0, 2)$
12. $(-3, 0), (-3, 6)$

13. **Reasoning**
Plot the points $A(6, 5), B(2, 5)$, and $C(6, 2)$. Find the distance between points $A$ and $B$. Find the distance between points $A$ and $C$. Use the distances and the Pythagorean theorem to find the distance between points $B$ and $C$. 

**ACTIVITY**
Investigating Algebra
11.5 Apply the Distance and Midpoint Formulas

Before
You used the Pythagorean theorem and its converse.

Now
You will use the distance and midpoint formulas.

Why?
So you can calculate distances traveled, as in Ex. 47.

Key Vocabulary
• distance formula
• midpoint
• midpoint formula

To find the distance between the points \( A(-2, -1) \) and \( B(3, 2) \), draw a right triangle, as shown. The lengths of the legs of the right triangle are as follows.

\[ AC = |3 - (-2)| = 5 \]
\[ BC = |-1 - 2| = 3 \]

You can use the Pythagorean theorem to find \( AB \), the length of the hypotenuse of the right triangle.

\[ (AB)^2 = (AC)^2 + (BC)^2 \]

Pythagorean theorem

\[ AB = \sqrt{(AC)^2 + (BC)^2} \]

Take positive square root of each side.

\[ AB = \sqrt{5^2 + 3^2} = \sqrt{34} \]

Substitute 5 for \( AC \) and 3 for \( BC \) and simplify.

This example suggests that you can find the distance between two points in a coordinate plane using the following formula, called the **distance formula**.

**KEY CONCEPT**

**The Distance Formula**

The distance \( d \) between any two points \( (x_1, y_1) \) and \( (x_2, y_2) \) is

\[ d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \]

**EXAMPLE 1**

Find the distance between two points

Find the distance between \((-1, 3)\) and \((5, 2)\).

Let \((x_1, y_1) = (-1, 3)\) and \((x_2, y_2) = (5, 2)\).

\[ d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \]

Distance formula

\[ = \sqrt{(5 - (-1))^2 + (2 - 3)^2} \]

Substitute.

\[ = \sqrt{6^2 + (-1)^2} = \sqrt{37} \]

Simplify.

The distance between the points is \(\sqrt{37}\) units.
**Example 2** Find a missing coordinate

The distance between \((3, -5)\) and \((7, b)\) is 5 units. Find the value of \(b\).

**Solution**

Use the distance formula with \(d = 5\). Let \((x_1, y_1) = (3, -5)\) and \((x_2, y_2) = (7, b)\). Then solve for \(b\).

\[
d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}
\]

Distance formula

\[
5 = \sqrt{(7 - 3)^2 + (b - (-5))^2}
\]

Substitute.

\[
5 = \sqrt{16 + b^2 + 10b + 25}
\]

Multiply.

\[
5 = \sqrt{b^2 + 10b + 41}
\]

Simplify.

\[
25 = b^2 + 10b + 41
\]

Square each side.

\[
0 = b^2 + 10b + 16
\]

Write in standard form.

\[
0 = (b + 2)(b + 8)
\]

Factor.

\[
b + 2 = 0 \quad \text{or} \quad b + 8 = 0
\]

Zero-product property

\[
b = -2 \quad \text{or} \quad b = -8
\]

Solve for \(b\).

The value of \(b\) is \(-2\) or \(-8\). 

**Guided Practice** for Examples 1 and 2

Find the distance between the points.

1. \((3, 0), (3, 6)\)
2. \((-2, 1), (2, 5)\)
3. \((6, -2), (-4, 7)\)
4. The distance between \((1, a)\) and \((4, 2)\) is 3 units. Find the value of \(a\). 

**Midpoint** The midpoint of a line segment is the point on the segment that is equidistant from the endpoints. You can find the coordinates of the midpoint of a line segment using the following formula, called the **midpoint formula**.

**Key Concept** For Your Notebook

**The Midpoint Formula**

The midpoint \(M\) of the line segment with endpoints \(A(x_1, y_1)\) and \(B(x_2, y_2)\) is

\[
M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)
\]

\[\text{Example 2} \quad \text{Find a missing coordinate}\]

The distance between \((3, -5)\) and \((7, b)\) is 5 units. Find the value of \(b\).

**Solution**

Use the distance formula with \(d = 5\). Let \((x_1, y_1) = (3, -5)\) and \((x_2, y_2) = (7, b)\). Then solve for \(b\).

\[
d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}
\]

Distance formula

\[
5 = \sqrt{(7 - 3)^2 + (b - (-5))^2}
\]

Substitute.

\[
5 = \sqrt{16 + b^2 + 10b + 25}
\]

Multiply.

\[
5 = \sqrt{b^2 + 10b + 41}
\]

Simplify.

\[
25 = b^2 + 10b + 41
\]

Square each side.

\[
0 = b^2 + 10b + 16
\]

Write in standard form.

\[
0 = (b + 2)(b + 8)
\]

Factor.

\[
b + 2 = 0 \quad \text{or} \quad b + 8 = 0
\]

Zero-product property

\[
b = -2 \quad \text{or} \quad b = -8
\]

Solve for \(b\).

The value of \(b\) is \(-2\) or \(-8\). 

**Guided Practice** for Examples 1 and 2

Find the distance between the points.

1. \((3, 0), (3, 6)\)
2. \((-2, 1), (2, 5)\)
3. \((6, -2), (-4, 7)\)
4. The distance between \((1, a)\) and \((4, 2)\) is 3 units. Find the value of \(a\). 

**Midpoint** The midpoint of a line segment is the point on the segment that is equidistant from the endpoints. You can find the coordinates of the midpoint of a line segment using the following formula, called the **midpoint formula**.

**Key Concept** For Your Notebook

**The Midpoint Formula**

The midpoint \(M\) of the line segment with endpoints \(A(x_1, y_1)\) and \(B(x_2, y_2)\) is

\[
M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)
\]
You and a friend are sightseeing in Washington, D.C. You are at the National Gallery of Art, and your friend is at the Washington Monument, as shown on the map. You want to meet at the landmark that is closest to the midpoint of your locations. At which landmark should you meet?

Solution

Your coordinates are (11, 3), and your friend’s coordinates are (2, 2). First, find the midpoint of your locations, which is

\[
\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) = \left( \frac{11 + 2}{2}, \frac{3 + 2}{2} \right) = (6.5, 2.5).
\]

Next, find the distance from the midpoint to the Smithsonian Institution, located at (7, 1), and to the Natural History Museum, located at (7, 3).

Distance to Smithsonian Institution: \(d = \sqrt{(6.5 - 7)^2 + (2.5 - 1)^2} \approx 1.58 \text{ units}\)

Distance to Natural History Museum: \(d = \sqrt{(6.5 - 7)^2 + (2.5 - 3)^2} \approx 0.71 \text{ unit}\)

You should meet at the Natural History Museum.
1. **VOCABULARY** Copy and complete: The point on a line segment that is equidistant from its endpoints is called the \( ? \) of the line segment.

2. ★ **WRITING** You want to know the distance between the points (3, 2) and (6, 8). Does it matter which point represents \((x_1, y_1)\) and which point represents \((x_2, y_2)\)？ Explain.

### FINDING DISTANCE

Find the distance between the two points.

3. (4, 8), (4, 7)  
4. (5, −9), (8, −9)  
5. (2, −2), (6, 1)  
6. (5, 1), (0, 3)  
7. (−4, 1), (3, −1)  
8. (2, 4), (−5, 0)  
9. (−6, 7), (2, 9)  
10. (−10, 8), (2, −3)  
11. (7, 5), (−12, −1)  
12. (4, 2, 5), (2, 5, −3)  
13. \((5, \frac{1}{2}), (−3, \frac{5}{2})\)  
14. \((-\frac{3}{4}, 2), (\frac{5}{4}, 4)\)

15. ★ **MULTIPLE CHOICE** What is the distance between \((4.5, 1)\) and \((-2.5, −5)\)?

   - A \( \sqrt{13} \)  
   - B \( \sqrt{24} \)  
   - C \( \sqrt{68.5} \)  
   - D \( \sqrt{85} \)

### FINDING MISSING COORDINATES

The distance \(d\) between two points is given. Find the value of \(b\).

16. \((0, b), (3, 1)\); \(d = 5\)  
17. \((13, −3), (b, 2)\); \(d = 13\)  
18. \((−9, −2), (b, 5)\); \(d = 7\)

19. \((b, −6), (−5, 2)\); \(d = 10\)  
20. \((−6, 8), (−1, b)\); \(d = \sqrt{29}\)  
21. \((b, −4), (4, 7)\); \(d = 11\sqrt{2}\)

### FINDING THE MIDPOINT

Find the midpoint of the line segment with the given endpoints.

22. \((0, 1), (8, 3)\)  
23. \((6, −3), (4, −7)\)  
24. \((−5, 0), (1, 14)\)  
25. \((11, −4), (−9, −4)\)  
26. \((−6, 6), (4, −4)\)  
27. \((−17, −8), (−5, −4)\)  
28. \((2, 7), (5, 3)\)  
29. \((−2, 3), (−2, −3)\)  
30. \((12, −5), (−12, 4)\)  
31. \((−15, −8), (−1, −1)\)  
32. \((18, −17), (12, −7)\)  
33. \((−50, −75), (8, 9)\)

34. ★ **MULTIPLE CHOICE** What is the midpoint of the line segment with endpoints \((2, 1)\) and \((4, 7)\)?

   - A \( (1, 3) \)  
   - B \( (1.5, 5.5) \)  
   - C \( (3, 4) \)  
   - D \( (4, 3) \)
ERROR ANALYSIS Describe and correct the error in finding the distance between \((-17, -2)\) and \((3, 8)\), and the midpoint of the line segment with endpoints \((-17, -2)\) and \((3, 8)\).

35. Distance:

\[
d = \sqrt{(3 - (-17))^2 - (8 - (-2))^2} = \sqrt{400 - 100} = \sqrt{300} = 10\sqrt{3}
\]

36. Midpoint:

\[
\left(\frac{3 - (-17)}{2}, \frac{8 - (-2)}{2}\right) = \left(10, 5\right)
\]

37. ★ MULTIPLE CHOICE What is the distance between point \(A\) and the midpoint of the line segment that joins points \(A\) and \(B\)?

- \(A\) \(\sqrt{17}\) units
- \(B\) \(3\sqrt{5}\) units
- \(C\) \(2\sqrt{17}\) units
- \(D\) \(\sqrt{117}\) units

FINDING ENDPOINTS The midpoint and an endpoint of a line segment are given. Find the other endpoint.

38. endpoint: \((1, 2)\) 39. endpoint: \((-2, -4)\) 40. endpoint: \((7, 5)\)

FINDING ENDPOINTS The midpoint and an endpoint of a line segment are given. Find the other endpoint.

38. endpoint: \((1, 2)\) 39. endpoint: \((-2, -4)\) 40. endpoint: \((7, 5)\)

41. \((3, 5), (3, -1), (-2, -1)\)
42. \((-3, -1), (1, 4), (-3, 0)\)
43. \((-5, -2), (0, -4), (-2, 3)\)
44. \((-2, 1), (-4, 3), (-8, -1)\)

45. ★ WRITING Explain how you can use the distance formula to verify that the midpoint of a line segment is equidistant from its endpoints.

46. CHALLENGE The midpoint of a line segment is \((0, 0)\). The line segment has a length of 2 units. Give three possible sets of endpoints for the line segment. Explain how you found your answer.

PROBLEM SOLVING

47. MULTI-STEP PROBLEM A rescue helicopter and an ambulance are both traveling from the dispatch center to the scene of an accident. The distance between consecutive grid lines represents 1 mile.

a. Find the distance that the ambulance traveled (red route).

b. How many times greater is the distance that the ambulance traveled than the distance that the helicopter traveled (blue route)?
48. **SUBWAY** A student is taking the subway to the public library. The student can get off the subway at one of two stops, as shown in the map. The distance between consecutive grid lines represents 0.25 mile. Which stop is closer to the library?

![Map of subway stops](image)

49. **ARCHAEOLOGY** Underwater archaeologists sometimes lay survey grids of the site they are studying. A sample survey grid is shown. The distance between consecutive grid lines represents 50 feet.

![Survey grid](image)

a. Which is shorter, the distance between the anchor and the sword or the distance between the anchor and the cup?

b. Which two objects are closest together? Which two objects are farthest apart?

50. **SHORT RESPONSE** The point of no return in aviation is the farthest point to which a plane can fly and still have enough fuel to return to its starting place or to fly to an alternative landing destination. After a plane passes the point of no return, it must fly to its planned destination. The distance between consecutive grid lines represents 50 nautical miles.

a. The flight path of a plane is from airport A to airport B. The plane is currently at the midpoint of the flight path. How far away is the plane from airport A? Round your answer to the nearest nautical mile.

b. The plane’s point of no return is calculated to be 200 nautical miles. Has the plane reached its point of no return? Explain.

51. **ROAD SIGN** Describe the quadrilateral formed by the sides of the road sign shown by answering the following questions.

- Are opposite sides parallel?
- Do the sides form right angles?
- Which sides, if any, are congruent?

![Road sign](image)
52. **CHALLENGE** A computer programmer is creating a baseball player’s strike zone for a video game, as shown. The strike zone is a rectangular region over home plate through which a ball must pass to be called a strike. In the animation, \( \overline{AB} \) is the top of the strike zone and lies on a horizontal line that passes through the midpoint of \( \overline{XY} \). The distance between grid lines represents 1 foot.

a. If the coordinates of \( X \) are \((4, 5.5)\) and the coordinates of \( Y \) are \((4, 3.5)\), what is the midpoint of \( \overline{XY} \)?

b. The coordinates of \( C \) are \((7, 2)\) and the coordinates of \( D \) are \((8.5, 2)\). Find the coordinates of point \( A \) and point \( B \).

c. What is the area of the strike zone in the animation?

---

**MIXED REVIEW**

Given that \( y \) varies directly with \( x \), use the specified values to write a direct variation equation that relates \( x \) and \( y \). (p. 253)

53. \( x = 10, y = 30 \)  
54. \( x = 81, y = 27 \)  
55. \( x = -6, y = -9 \)

56. \( x = 12, y = -1.5 \)  
57. \( x = -11, y = -11 \)  
58. \( x = \frac{2}{3}, y = 6 \)

Graph the function.

59. \( y = 4^x \) (p. 520)  
60. \( y = 3 \cdot 4^x \) (p. 520)  
61. \( y = (0.5)^x \) (p. 531)

62. \( y = 2 \cdot (0.5)^x \) (p. 531)  
63. \( y = x^2 - 6 \) (p. 628)  
64. \( y = 2x^2 - x + 8 \) (p. 635)

---

**QUIZ for Lessons 11.4–11.5**

Let \( a \) and \( b \) represent the lengths of the legs of a right triangle, and let \( c \) represent the length of the hypotenuse. Find the unknown length. (p. 737)

1. \( a = 6, c = 10 \)  
2. \( b = 2, c = 6 \)  
3. \( a = 4, b = 7 \)

Find the unknown lengths. (p. 737)

4. \[ \begin{align*} 4 + x & \end{align*} \]  
5. \[ \begin{align*} 2x & \end{align*} \]  
6. \[ \begin{align*} 3x - 2 & \end{align*} \]

Find the distance between the two points. (p. 744)

7. \((7, 2), (7, 5)\)  
8. \((-1, -3), (4, -3)\)  
9. \((0, 0), (-6, 9)\)

Find the midpoint of the line segment with the given endpoints. (p. 744)

10. \((0, 5), (-6, 3)\)  
11. \((8, -1), (2, -7)\)

12. \((-5, -3), (5, -3)\)

13. \((0, 6), (1.5, 4)\)  
14. \((2.5, -3), (0.5, 6)\)

15. \[ \left( \frac{1}{4}, \frac{3}{4} \right) \]  
[ \left( \frac{1}{4}, \frac{5}{4} \right) \]
Another Way to Solve Example 4, page 746

**MULTIPLE REPRESENTATIONS** In Example 4 on page 746, you saw how to solve a problem about finding a meeting place by using the midpoint and distance formulas. You can also solve the problem by folding a map and using a compass.

**SIGHTSEEING** You and a friend are sightseeing in Washington, D.C. You are at the National Gallery of Art, and your friend is at the Washington Monument, as shown on the map. You want to meet at the landmark that is closest to the midpoint of your locations. At which landmark should you meet?

**METHOD 1**  
Folding a map and using a compass An alternative approach is to fold a map and use a compass. First, draw a line connecting your location to your friend’s location. Then fold the map so that your locations coincide. The point where the line connecting your locations is folded represents the midpoint. Place the point of your compass at the midpoint. Adjust the opening of the compass to match the distance between the midpoint and the apparent closest landmark. Swing the compass to see if the other landmark is closer.

Because the Smithsonian lies outside the circle, the Natural History Museum is closer to the midpoint of your locations.

**PRACTICE**

**1. WHAT IF?** In the problem above, suppose your friend is at the White House.
   a. At which landmark should you meet?
   b. Suppose you can walk directly to the landmark in part (a). If the distance between consecutive grid lines represents 0.06 mile, how far do you have to walk?

**2. MAPS** A student makes a map of a town in which the student’s house is located at (1, 2) and a friend’s house is located at (8, 5). A grocery store is located at (5, 3), and a shoe store is located at (3, 4). The student and the friend want to meet at the store that is closer to the midpoint between their houses. At which store should they meet? Solve this problem using two methods.
1. SHORT RESPONSE  Construction workers are building a staircase for a house. Use the drawing of the staircase to answer the following questions. Round each answer to the nearest inch.

![Staircase diagram]

a. Find the distance \( d \) between the edges of two consecutive steps.

b. Suppose the workers install a handrail that is as long as the distance between the front edge of the bottom step and the front edge of the top step. How long is the handrail? Explain.

2. MULTI-STEP PROBLEM  At the start of a football game, the kicker on one team must kick the ball to the opposing team. To position himself for the starting kick, the kicker places a football on a tee, walks 8 yards behind the tee, then 6 yards to his left.

![Football diagram]

a. What is the kicker's distance (in yards) from the football?

b. The kicker takes 11 strides to kick the ball. What is his average stride length? Round your answer to the nearest tenth of a foot.

3. GRIDDED ANSWER  You go on a hiking trip. You walk 2 miles directly east and then 4 miles directly north. If you could walk in a straight path back to your starting point, how far would you have to walk? Round your answer to the nearest tenth of a mile.

4. SHORT RESPONSE  A map of a town is shown. A student is at the school. The student's friend is at the stadium. They want to meet at the place that is closest to the midpoint of their locations. At which location should they meet? Explain how you found your answer.

![Map of town]

5. OPEN-ENDED  City planners want to build a rectangular park. They want the park to have a straight path that is 2500 feet long and connects opposite corners of the park. What are three possibilities for the length and width of the park?

6. EXTENDED RESPONSE  One ferry makes round trips shown in red. A second ferry makes round trips shown in blue. The distance between consecutive grid lines represents 1 mile.

![Ferry routes]

a. Find the distance (in miles) of a round trip for the first ferry.

b. Find the distance (in miles) of a round trip for the second ferry.

c. Each stop the ferries make takes 10 minutes. The first ferry travels at 20 miles per hour, and the second ferry travels at 16 miles per hour. The ferries leave Shore City at the same time each day. When will they be back at the city at the same time? Explain.
**BIG IDEAS**

**Graphing Square Root Functions**
You can graph a square root function \( y = a\sqrt{x - h} + k \) and compare its graph with the graph of the parent function, \( y = \sqrt{x} \), based on the constants \( a \), \( h \), and \( k \).

<table>
<thead>
<tr>
<th>Constant</th>
<th>Comparison of graphs</th>
</tr>
</thead>
</table>
| \( a \)  | • When \( a > 0 \), the graph is a vertical stretch or shrink of the parent graph.  
           | • When \( a < 0 \), the graph is a vertical stretch or shrink with a reflection in the \( x \)-axis of the parent graph. |
| \( h \)  | The graph is a horizontal translation of the parent graph. |
| \( k \)  | The graph is a vertical translation of the parent graph. |

**Using Properties of Radicals in Expressions and Equations**
You can use the properties of radicals to simplify radical expressions and to solve radical equations.

| Product property of radicals | \( \sqrt{ab} = \sqrt{a} \cdot \sqrt{b} \) where \( a \geq 0 \) and \( b \geq 0 \) |
| Quotient property of radicals | \( \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}} \) where \( a \geq 0 \) and \( b > 0 \) |

**Working with Radicals in Geometry**
You can use radicals to solve problems involving the following geometric theorems and formulas.

| Pythagorean theorem | If a triangle is a right triangle, then the sum of the squares of the lengths of the legs, \( a \) and \( b \), equals the square of the length of the hypotenuse \( c \).  
                      | \( a^2 + b^2 = c^2 \) |
| Converse of Pythagorean theorem | If a triangle has side lengths \( a \), \( b \), and \( c \) such that \( a^2 + b^2 = c^2 \), then the triangle is a right triangle. |
| Distance formula | \( d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \) |
| Midpoint formula | \( M = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \) |
# REVIEW KEY VOCABULARY

- radical expression, p. 710
- radical function, p. 710
- square root function, p. 710
- parent square root function, p. 710
- simplest form of a radical expression, p. 719
- rationalizing the denominator, p. 721
- radical equation, p. 729
- extraneous solution, p. 730
- hypotenuse, legs of a right triangle, p. 737
- Pythagorean theorem, p. 737
- distance formula, p. 744
- midpoint, midpoint formula, p. 745

## VOCABULARY EXERCISES

1. Describe how the graph of the function \( y = 3\sqrt{x} \) compares with the graph of the parent square root function.

2. Describe the steps you would take to rationalize the denominator of a radical expression.

Tell which theorem or formula you would use to complete the exercise.

3. Tell whether a triangle with side lengths 2, 4, and 6 is a right triangle.

4. The point \((b, 4)\) is 10 units away from the point \((5, 10)\). Find \(b\).

## REVIEW EXAMPLES AND EXERCISES

Use the review examples and exercises below to check your understanding of the concepts you have learned in each lesson of Chapter 11.

### 11.1 Graph Square Root Functions

**Example**

Graph the function \( y = \sqrt{x - 3} \) and identify its domain and range. Compare the graph with the graph of \( y = \sqrt{x} \).

To graph the function, make a table, plot the points, and draw a smooth curve through the points. The domain is \( x \geq 3 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>0</td>
<td>1</td>
<td>1.4</td>
<td>1.7</td>
</tr>
</tbody>
</table>

The range is \( y \geq 0 \). The graph of \( y = \sqrt{x - 3} \) is a horizontal translation (of 3 units to the right) of the graph of \( y = \sqrt{x} \).

**Exercises**

Graph the function and identify its domain and range. Compare the graph with the graph of \( y = \sqrt{x} \).

5. \( y = -2\sqrt{x} \)  
6. \( y = \sqrt{x} + 7 \)  
7. \( y = \sqrt{x + 7} \)
11.2 Simplify Radical Expressions

**Example**

Simplify $7\sqrt{5} - \sqrt{45}$.

$7\sqrt{5} - \sqrt{45} = 7\sqrt{5} - \sqrt{9 \cdot 5}$

$= 7\sqrt{5} - 3\sqrt{5}$

$= 4\sqrt{5}$

**EXERCISES**

Simplify the expression.

8. $\sqrt{98}$
9. $\sqrt{121x^3}$
10. $\sqrt{7} \cdot \sqrt{21}$
11. $\sqrt{7x} \cdot \sqrt{x}$
12. $\frac{5}{\sqrt{x^2}}$
13. $\frac{2}{\sqrt{5}}$
14. $3\sqrt{2} - \sqrt{128}$
15. $\sqrt{2}(7 - \sqrt{6})$

16. **GEOMETRY** The lateral surface area $L$ of a square pyramid with height $h$ and base length $l$ is given by $L = 2l\sqrt{0.25l^2 + h^2}$. Find $L$ (in square feet) for a square pyramid that has a height of 4 feet and a base length of 4 feet.

11.3 Solve Radical Equations

**Example**

Solve $\sqrt{x} + 90 = x$.

$\sqrt{x} + 90 = x$

$\left(\sqrt{x} + 90\right)^2 = x^2$

$x + 90 = x^2$

$0 = x^2 - x - 90$

$0 = (x - 10)(x + 9)$

$x - 10 = 0$ or $x + 9 = 0$

$x = 10$ or $x = -9$

Checking 10 and −9 in the original equation shows that −9 is an extraneous solution. The only solution of the equation is 10.

**EXERCISES**

Solve the equation. Check for extraneous solutions.

17. $\sqrt{x} - 28 = 0$
18. $8\sqrt{x} - 5 + 34 = 58$
19. $\sqrt{5x} - 3 = \sqrt{x} + 17$
20. $\sqrt{5x} + 6 = 5$
21. $\sqrt{x} + 36 = 0$
22. $x = \sqrt{2} - x$
Example

Find the unknown length for the triangle shown.

\[ a^2 + b^2 = c^2 \]  
Pythagorean theorem

\[ 6^2 + b^2 = 11^2 \]
Substitute 6 for \( a \) and 11 for \( c \).

\[ 36 + b^2 = 121 \]
Simplify.

\[ b^2 = 85 \]
Subtract 36 from each side.

\[ b = \sqrt{85} \]
Take positive square root of each side.

Exercises

Let \( a \) and \( b \) represent the lengths of the legs of a right triangle, and let \( c \) represent the length of the hypotenuse. Find the unknown length.

23. \( a = 7, b = 13 \)  
24. \( a = 10, c = 21 \)  
25. \( a = 8, c = 11 \)

26. \( a = 9, b = 17 \)  
27. \( b = 4, c = 15 \)  
28. \( b = 6, c = 6.5 \)

29. Reflecting Pool: The Reflecting Pool in front of the Lincoln Memorial in Washington, D.C., is rectangular with a length of 2029 feet and a width of 167 feet. To the nearest foot, what is the length of a diagonal of the Reflecting Pool?

Example

Find the distance between \((-3, 8)\) and \((5, -12)\).

Let \((x_1, y_1) = (-3, 8)\) and \((x_2, y_2) = (5, -12)\).

\[ d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \]
Distance formula

\[ d = \sqrt{(5 - (-3))^2 + (-12 - 8)^2} \]
Substitute.

\[ d = \sqrt{64} = 4\sqrt{29} \]
Simplify.

Exercises

Find the distance between the two points.

30. \((-1, -3), (9, -13)\)  
31. \((-8, -4), (0, 2)\)  
32. \((7, 1), (4, -0.25)\)

Find the midpoint of the line segment with the given endpoints.

33. \((-2, -4), (9, -4)\)  
34. \((-8, 0), (-8, 2)\)  
35. \((6, 1), (4, -5)\)

36. Islands: On a coordinate grid, an island is located at \((1, 6)\). Another island is located at \((4, 9)\). What is the distance between the islands if the distance between consecutive grid lines represents 2 miles?
Graph the function and identify its domain and range. Compare the graph with the graph of \( y = \sqrt{x} \).

1. \( y = 3\sqrt{x} \)
2. \( y = -\sqrt{x} \)
3. \( y = \sqrt{x - 5} \)
4. \( y = -\sqrt{x} - 1 + 4 \)

Simplify the expression.

5. \( \sqrt{72m^6} \)
6. \( \sqrt{8z^3} \cdot \sqrt{6z^3} \)
7. \( \frac{20}{\sqrt{3n^3}} \)
8. \( 7\sqrt{6} - 2\sqrt{12} + \sqrt{24} \)
9. \( \sqrt{3(7 - \sqrt{15})} \)
10. \( (8 - \sqrt{7})(1 + \sqrt{7}) \)

Solve the equation. Check for extraneous solutions.

11. \( \sqrt{x} = 8 \)
12. \( \sqrt{x} + 5 - 6 = -2 \)
13. \( -4\sqrt{3x} - 6 = 30 \)
14. \( \sqrt{5x - 11} = \sqrt{x} \)
15. \( \sqrt{x} + 7 = \sqrt{2x - 3} \)
16. \( x = \sqrt{12} - x \)

Find the unknown lengths.

17. \( \triangle \) with \( \sqrt{58} \) and \( x + 4 \)
18. \( \triangle \) with \( x + 5 \) and \( x - 5 \)
19. \( \triangle \) with \( \sqrt{x} \) and \( 2x - 5 \)

Tell whether the triangle with the given side lengths is a right triangle.

20. 8, 16, 32
21. 11, 60, 61
22. 7.5, 10, 12.5

Find the distance between the given points. Then find the midpoint of the line segment whose endpoints are the given points.

23. \((6, 6), (9, 10)\)
24. \((-8, 7), (4, 3)\)
25. \((5, -\frac{3}{2}), (-2, \frac{9}{2})\)

26. **LADDERS** A ladder that is 25 feet long is placed against a house. The bottom of the ladder is 10 feet from the base of the house. How far up the house does the ladder reach? Round your answer to the nearest tenth of a foot.

27. **BIRD HOUSES** The front view of a bird house is shown. Find the height of the house to the nearest tenth of a foot.

28. **LACROSSE** Two lacrosse players are playing on a field, as shown. The distance between consecutive grid lines represents 2 meters.
   a. How far is each player from the ball? Round your answer to the nearest tenth of a meter.
   b. Both players start running toward the ball.
      Player A can run at a rate of 6 meters per second.
      Player B can run at a rate of 7 meters per second.
      Who will reach the ball first?
MULTIPLE CHOICE QUESTIONS

If you have difficulty solving a multiple choice problem directly, you may be able to use another approach to eliminate incorrect answer choices and obtain the correct answer.

**Problem 1**

What is the solution of the equation $\sqrt{3 - x} = 2x$?

A. $1, -\frac{3}{4}$  
B. $-1, \frac{3}{4}$  
C. $-1$  
D. $\frac{3}{4}$

**Method 1**

**Solve directly** Solve the radical equation and check for extraneous solutions.

**Step 1** Solve the radical equation.

\[
\sqrt{3 - x} = 2x
\]

\[
3 - x = 4x^2
\]

\[
0 = 4x^2 + x - 3
\]

\[
0 = (4x - 3)(x + 1)
\]

\[
x = \frac{3}{4} \text{ or } x = -1
\]

**Step 2** Check $\frac{3}{4}$ and $-1$ in the original equation.

**Check $\frac{3}{4}$**:

\[
\sqrt{3 - \frac{3}{4}} \geq 2\left(\frac{3}{4}\right)
\]

\[
\frac{3}{2} = \frac{3}{2} \checkmark
\]

Solution checks.

**Check $-1$**:

\[
\sqrt{3 - (-1)} \geq 2(-1)
\]

\[
2 = -2 \times
\]

Solution does not check.

The only solution is $\frac{3}{4}$.

The correct answer is D. (A) (B) (C) (D)

**Method 2**

**Eliminate choices** Substitute the values given in each answer choice for $x$ in the equation.

**Choice A**: $1, -\frac{3}{4}$

\[
\sqrt{3 - 1} \geq 2(1) \quad \sqrt{3 - \left(-\frac{3}{4}\right)} \geq -\frac{3}{4}
\]

\[
\sqrt{2} \times \quad \frac{\sqrt{15}}{2} = -\frac{3}{4} \times
\]

The values do not check, so choice A can be eliminated.

**Choice B**: $-1, \frac{3}{4}$

\[
\sqrt{3 - (-1)} \geq 2(-1) \quad \sqrt{3 - \frac{3}{4}} \geq 2\left(\frac{3}{4}\right)
\]

\[
2 = -2 \times \quad \frac{3}{2} = \frac{3}{2} \checkmark
\]

Because $-1$ does not check, both choice B and choice C can be eliminated.

**Choice D**: $\frac{3}{4}$

You know that $\frac{3}{4}$ is a solution from checking the values in choice B.

The only solution is $\frac{3}{4}$.

The correct answer is D. (A) (B) (C) (D)
A carpenter is building a wooden bench and wants to be sure that the back and the seat make a right angle. The back is 24 inches tall, and the seat is 18 inches deep. What should the distance from the front of the seat to the top of the back be?

A) 6 inches  
B) 15.9 inches  
C) 30 inches  
D) 42 inches

**Method 1**

**Solve directly** Use the Pythagorean theorem to find the unknown distance.

**Step 1** Identify the known values by drawing a diagram.

![Diagram](a = 24 in., b = 18 in., c)

**Step 2** Substitute the values of \(a\) and \(b\) and solve for \(c\).

\[a^2 + b^2 = c^2\]
\[18^2 + 24^2 = c^2\]
\[900 = c^2\]
\[c = 30\]

The distance from the front of the seat to the top of the back should be 30 inches.

The correct answer is C.  A  B  C  D

**Method 2**

**Eliminate choices** You can eliminate choices by using the converse of the Pythagorean theorem. Check to see if the value given in each answer choice could represent the length of the hypotenuse of a right triangle with leg lengths of 18 and 24.

**Choice A:** 6 inches

\[18^2 + 24^2 = 6^2\]
\[900 ≠ 36 \quad \times\]

**Choice B:** 15.9 inches

\[18^2 + 24^2 = 15.9^2\]
\[900 = 252.81 \quad \times\]

**Choice C:** 30 inches

\[18^2 + 24^2 = 30^2\]
\[900 = 900 \quad \checkmark\]

The correct answer is C.  A  B  C  D

**Practice**

Explain why you can eliminate the highlighted answer choice.

1. What is the solution of the equation \(\sqrt{20} - x = x\)?

   A) 4, -5  
   B) -4, 5  
   C) X -5  
   D) 4

2. Which of the following represents the side lengths of a right triangle?

   A) 1, 2, 3  
   B) 6, 8, 14  
   C) 12, 13, 15  
   D) 8, 15, 17

3. A side view of a wheelchair ramp can be represented by the hypotenuse of a right triangle. The triangle has a length of 24 feet and a height of 2 feet. To the nearest tenth, how long is the ramp?

   A) 22 feet  
   B) 23.9 feet  
   C) 24.1 feet  
   D) 26 feet
1. What is the solution of the equation \( \sqrt{2x + 8} = x \)?
   - A) -2, 4
   - B) 24, 2
   - C) -2
   - D) 4

2. Which expression is equivalent to \( \sqrt{4x^2y} \cdot \sqrt{y} \)?
   - A) 4xy
   - B) 2xy
   - C) 2x\sqrt{y}
   - D) 4x\sqrt{y}

3. The graph of which function is shown?
   - A) \( y = \sqrt{x} - 3 \)
   - B) \( y = 2\sqrt{x} - 3 \)
   - C) \( y = 2\sqrt{x} + 3 \)
   - D) \( y = 2\sqrt{x} - 3 \)

4. How does the graph of \( y = \sqrt{x} + 3 \) compare with the graph of \( y = \sqrt{x} \)?
   - A) It is a vertical stretch by a factor of 3 of the graph of \( y = \sqrt{x} \).
   - B) It is a vertical translation of 3 units up of the graph of \( y = \sqrt{x} \).
   - C) It is a vertical translation of 3 units down of the graph of \( y = \sqrt{x} \).
   - D) It is a horizontal translation of 3 units to the right of the graph of \( y = \sqrt{x} \).

5. Which expression represents the length of a diagonal of the rectangle?
   - A) \( 3x^2 + 100 \)
   - B) \( 9x^2 + 100 \)
   - C) \( \sqrt{9x^2 + 100} \)
   - D) \( \sqrt{3x^2 + 100} \)

6. What is the solution of the equation \( \sqrt{x + 3} = x - 9 \)?
   - A) -13, -6
   - B) 13, 6
   - C) 13
   - D) 6

7. The table below represents which function?

<table>
<thead>
<tr>
<th>x</th>
<th>4</th>
<th>5</th>
<th>8</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>5</td>
<td>8</td>
<td>11</td>
<td>14</td>
</tr>
</tbody>
</table>

   - A) \( y = 3\sqrt{x} - 4 + 5 \)
   - B) \( y = 3\sqrt{x} - 3 + 8 \)
   - C) \( y = 3\sqrt{x} - 4 + 8 \)
   - D) \( y = 3\sqrt{x} - 3 + 5 \)

In Exercises 8 and 9, use the map of the college campus shown.

8. Which of the following pairs of buildings are closest together?
   - A) Dorm A and the library
   - B) Dorm B and the gym
   - C) The gym and the library
   - D) Dorm B and the library

9. A student who lives in dorm A forgets a book at the library. The student jogs at a rate of 6 miles per hour from the dorm straight to the library and back. The distance between consecutive grid lines represents 0.1 mile. To the nearest tenth of an hour, how long does it take the student to jog to the library and back?
   - A) 0.1 hour
   - B) 0.2 hour
   - C) 0.3 hour
   - D) 0.4 hour
### GRIDDED ANSWER

10. A line segment has endpoints (8, 0) and (14, 8). What is the distance from either endpoint to the midpoint?

11. A right triangle has a hypotenuse of 18 centimeters. The length of one leg is 8 centimeters. To the nearest tenth of a centimeter, how long is the other leg?

12. The leg lengths of a triangle are shown.

![Diagram of a triangle with sides labeled $\sqrt{3x - 5}$ and $x + 4$.]

To the nearest tenth, what is the length of the hypotenuse when $x = 3$?

13. A rectangular table is 7 feet long and 3.5 feet wide. To the nearest tenth of a foot, what is the length of a diagonal of the table?

14. What is the solution of the equation $2\sqrt{x} = 3\sqrt{8} - \sqrt{2}$? Write your answer as a decimal.

### SHORT RESPONSE

15. A person lives in Glenville. The person’s friend lives in Newport. The friends decide to meet at the city that is closer to the midpoint of their locations. In which city should they meet? Explain.

16. The front view of a shed is shown. Roofers are going to replace the tin roof of the shed.

![Diagram of a shed with dimensions 10 ft, 12 ft, and 15 ft.] If the depth of the shed is 20 feet, how many square feet of tin will the roofers need? Explain.

### EXTENDED RESPONSE

17. For the period 1900–2000, the life expectancy at birth $e$ (in years) for people born in the United States can be modeled by the function $e = 3.66\sqrt{t} + 40.6$ where $t$ is the number of years since 1900.

   a. Graph the function and identify its domain and range.

   b. In what year was the life expectancy at birth about 75 years? Explain.

   c. During which decade (1900–1909, 1910–1919, and so on) did life expectancy increase the most? Explain.

18. A planet’s mean radius $r$ (in meters) is given by $r = \sqrt{\frac{6.67 \times 10^{-11}M}{a}}$ where $a$ is the planet’s acceleration due to gravity (in meters per second squared) and $M$ is the planet’s mass (in kilograms).

   a. For Earth, the value of $a$ is about 9.8 meters per second squared, and the value of $M$ is about 5.98 $\times$ 10$^{24}$ kilograms. Find the value of $r$.

   b. For Jupiter, the value of $a$ is about 24.8 meters per second squared, and the value of $M$ is about 1.9 $\times$ 10$^{27}$ kilograms. Find the value of $r$.

   c. Jupiter’s mass is about 300 times Earth’s mass, and Jupiter’s acceleration due to gravity is about 2.5 times that of Earth’s. If you multiply $M$ by 300 and $a$ by 2.5, what happens to the value of $r$? Compare your answer with your results from parts (a) and (b).